CHAPTER SEVEN

Duration and Convexity

Duration

What is duration? Duration is a measure of the average life of a security. More specifically, it is the weighted average term-to-maturity of the security's cash flows. Mathematically, it is:

\[ t_1 \times PVCF_1 + t_2 \times PVCF_2 + t_3 \times PVCF_3 + \ldots + t_n \times PVCF_n \]

\[
\text{Duration} = \frac{1}{k} \times \text{PVTCF}
\]

where

- \( PVCF_t \) = the present value of the cash flow in period \( t \) discounted at the yield-to-maturity.
- \( PVTCF \) = the total present value of the cash flow of the security determined by the yield-to-maturity, or simply the price of the security.
- \( k \) = number of payments per year.

The following is an example of duration\(^1\):

<table>
<thead>
<tr>
<th>Period (t)</th>
<th>Cash flow</th>
<th>PVCF</th>
<th>( t \times PVCF_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$4.0</td>
<td>3.8462</td>
<td>3.8462</td>
</tr>
<tr>
<td>2</td>
<td>4.0</td>
<td>3.6982</td>
<td>7.3964</td>
</tr>
<tr>
<td>3</td>
<td>4.0</td>
<td>3.5560</td>
<td>10.6680</td>
</tr>
<tr>
<td>4</td>
<td>4.0</td>
<td>3.4192</td>
<td>13.6769</td>
</tr>
<tr>
<td>5</td>
<td>4.0</td>
<td>3.2877</td>
<td>16.4385</td>
</tr>
<tr>
<td>6</td>
<td>4.0</td>
<td>3.1613</td>
<td>18.9675</td>
</tr>
<tr>
<td>7</td>
<td>4.0</td>
<td>3.0397</td>
<td>21.2777</td>
</tr>
<tr>
<td>8</td>
<td>4.0</td>
<td>2.9228</td>
<td>23.3821</td>
</tr>
<tr>
<td>9</td>
<td>4.0</td>
<td>2.8103</td>
<td>25.2931</td>
</tr>
<tr>
<td>10</td>
<td>104.0</td>
<td>70.2586</td>
<td>702.5867</td>
</tr>
</tbody>
</table>

Total

| 100.0000 | 843.5331 |

Macaulay duration = 843.5331/(2 x 100) = 4.2177

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\(^1\) This example is adopted from “Price Volatility Characteristics of Fixed Income Securities” by Frank J Fabozzi, Mark Pitts and Ravi E Dattatreya.
This measure of duration is referred to as Macaulay's duration, which is named after Frederick Macaulay, who first computed it in 1938 in his article, "Some Theoretical Problems Suggested by the Movement of Interest Rates, Bond Yields, and Stock in the U.S. since 1856".

Duration is a single number that is measured in units of time, e.g. months or years. For securities that make only one payment at maturity, such as zero coupon bonds, duration equals term-to-maturity.

**Use of Duration**

Duration is a very useful measure. First, it converts a very complex cash flow structure into a single number that represents the rate sensitivity of the financial instrument. Second, it can be used in a formula to calculate the change in the security's market value that occurs as a result of a change in market rate. Third, it can also be used to measure the overall rate sensitivity of a portfolio of instruments or the entire asset liability structure.

The relationship between Macaulay’s duration and security price volatility is:

\[
\text{Percentage change in price} = -\frac{1}{1 + \text{yield/k}} \times \text{Macaulay’s duration} \times \text{yield change} \times 100 \quad (1)
\]

For easy calculation, academics and market practitioners usually combine the first two expressions on the right-hand side into one term and call it “modified duration”:

\[
\text{Modified duration} = \frac{\text{Macaulay’s duration}}{1 + \text{yield/k}} \quad (2)
\]

The relationship can then be expressed as follows:

\[
\text{Percentage change in price} = -\text{modified duration} \times \text{yield change} \times 100 \quad (3)
\]

To illustrate this relationship, consider a change in yield (change in market rate) from 8.00 percent to 9.00 percent for the bond described earlier:

\[
\text{Modified duration} = \frac{4.2177}{1 + 0.8/2} = 4.0555 \quad (4)
\]

\[
\text{Change in price} = -4.0555 \times (+0.01) \times 100 = -4.0555 \text{ percent} \quad (5)
\]
In other words, when the interest rate increases from 8.00 percent to 9.00 percent, the bond originally sold for 100, is now priced at 95.9445, a decrease of 4.0555 percent.

Equation (3) is quite accurate for small changes in yields, but is only an approximation for large changes in yields. This will be discussed later in this chapter.

From the above example and the previous illustration, we can see how duration applies to bond pricing analysis. (We constrain our discussion to option-free securities.)

First, we see from the illustration that we have converted a 5-year bond with 10 cash flows into a single number - a modified duration of 4.0555. What does it mean? It means that if the interest rate increases by, say one percent, the price of the security decreases by 4.0555 percent; or if the interest rate decreases by one percent, the price of the security increases by 4.0555 percent. Therefore, this modified duration of 4.0555 represents the interest rate sensitivity of this 5-year bond.

Second, as we can see in equations (3) and (5), the percentage change in bond price due to a change in interest rate can be easily calculated.

Third, by using the duration concept, we can perform some forms of asset/liability management. We can match the durations between assets and liabilities portfolios to create a "duration-matched portfolio". Or we can construct an" immunised portfolio" that provides assured returns over a target holding period.

In adddition, because modified duration is the percentage change in the price of a security given a change in yields, and a change in yields is a measure of the change in the bond market, modified duration actually plays a similar role of measuring market risk for bonds that beta plays for stocks.

Although the duration concept is simple and easy to apply in bond pricing analysis, it also has its shortcomings. As we mentioned earlier, duration is quite accurate in calculating changes in price for small changes in yields. For large changes in yields, duration only provides an approximation, and we have to rely on the concept of convexity for more accurate estimations.
**Convexity**

The true relationship between a change in price and a change in interest rate for option-free securities is not linear. The following graph demonstrates this relationship:

![Convexity Graph](image)

**Yield**

The actual price of the security is the curve AA', and the straight line BB' which is tangent to the AA' curve represents the duration of the security. Therefore, as the graph shows, the straight line approximation creates errors that grow with the magnitude of the interest rate changes. For small changes in yield, e.g. from Y to Y₂ or Y₃, duration does a good job in estimating the actual price. But for larger changes, e.g. from Y to Y₁ or Y₄, duration becomes less accurate and the errors become larger. Convexity is what we use to correct this problem.

Mathematically, modified duration is the first order derivative of bond price while convexity is the second order derivative. Convexity measures the rate of change of duration:

\[
\frac{t₁ \times (t₁+1) \times PVCF₁^2 + t₂ \times (t₂+1) \times PVCF₂^2 + \ldots + tₙ \times (tₙ+1) \times PVCFₙ^2}{(1 + \text{yield}/k)^2 \times (k^2 \times PVTCF)}
\]

Convexity = \[\frac{\text{rate of change of duration}}{(1 + \text{yield}/k)^2 \times (k^2 \times PVTCF)}\]  \hspace{1cm} (6)

The convexity of the security in our previous illustration is 20.1886. (following the calculation format illustrated on Page 32.)

The price change that is due to the curvature is shown in the following formula:
Percentage change in price due to convexity =
\[
\frac{1}{2} \times \text{convexity} \times (\text{yield change})^2 \times 100
\]  
(7)

Now we can put things together. For a 100 basis point change in yield from 8.00 percent to 7.00 percent, the true theoretical price, taking into consideration both duration and convexity, is:

\[
\begin{align*}
\text{Percentage change in price due to duration} & = -4.0555 \times (-0.01) \times 100 = 4.0555\% \\
\text{Percentage change in price due to convexity} & = \frac{1}{2} \times 20.1886 \times (-0.01)(-0.01) \times 100 \\
& = 0.1009\% \\
\text{Total percentage change in price} & = 4.0555\% + 0.1009\% \\
& = 4.1564\%
\end{align*}
\]

As you can see, when the interest rate decreases, the convexity actually *accelerates* the price appreciation. Further, the change in price due to convexity is largely determined by the change in interest rate. If the yield change is small, the square of yield change is negligible. If the yield change is large, this term can be significant and thus affects the outcome of equation (7).

Similarly, the true theoretical price change if the interest rate increases from 8.00 percent to 9.00 percent can be calculated as follows:

\[
\begin{align*}
\text{Percentage change in price due to duration} & = -4.0555 \times (+0.01) \times 100 = -4.0555\% \\
\text{Percentage change in price due to convexity} & = \frac{1}{2} \times 20.1886 \times (+0.01)(+0.01) \times 100 \\
& = 0.1009\% \\
\text{Total percentage change in price} & = -4.0555\% + 0.1009\% = -3.9546\%
\end{align*}
\]

When rate increases, the convexity actually *decelerates* the price depreciation. As shown on the graph on page 35, the tangent line lies below the theoretical price line. This implies that duration always underestimates the true security price.

All we are doing here is to figure out what the true theoretical prices are. In reality, observed market prices may be different from the theoretical prices because of bid-ask spread, demand and supply, market liquidity, market sentiment, etc. Asset pricing is more an art than an exact science and all mathematical models are just tools to facilitate the pricing analysis. I had once tried to confirm the market value of a multi-billion dollar, illiquid and below-investment-grade bond (or junk bond) portfolio in order to determine the viability of the institution who held these bonds. A world-famous junk bond consultant group was called in to do the evaluation. After days of hard work the consultants delivered their final analysis. The market values of most of these bonds given by the consultants ranged between 55 and 85 percent of par. The consultants said that if everything (including the economy, market sentiment of the company, market sentiment of junk bonds, interest rates, management’s ability, etc.) went well, the price could be 85, and if everything did not go well, the price
could be 55. In other words, the user of the analysis had to make a lot of subjective assumptions in order to come to a definitive conclusion.

Duration and convexity are very useful in bond pricing analysis. They are also useful tools in trading and derivatives activities. Traders often use something called price value basis point (PVBP) to measure the sensitivity of risk of their positions. PVBP uses the duration concept to measure the change in price of a security if rates move one basis point. This is a very useful tool for traders to measure their position sensitivity. However, the shortcoming of this measure (and all other duration measures) is that it assumes a parallel shift of the yield curve which usually does not happen in the real world. Therefore, PVBP is more useful for traders who trade short-term instruments.

Dr. Sam Srinivasulu of Michigan University has drawn an excellent analogy regarding duration, convexity, delta and gamma. He said, “duration (or delta) is the speed, and convexity (or gamma) is the acceleration." It is amazing to see how those bright people connect science and finance together. So we should not be surprised to see that more and more rocket scientists are now invading our finance territories.