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ASSESSING THE RISK OF MULTIPLE DEFAULTS IN THE BANKING SYSTEM

Key Points:

- *Among the various risks faced by banks that may have a systemic impact on the banking system, the credit risk has long been one of the concerns of central banks. This paper illustrates how to assess the credit risk of the banking system in Hong Kong by constructing an index of multiple default risk from a structural approach, using accounting information and up-to-date market-based information such as equity prices.*

- *In addition to an aggregate measure of individual banks' default probabilities, central banks also use an indicator of multiple defaults to monitor the systemic risk in the banking system. By incorporating asset correlations between banks, the multiple default risk index derived in this paper is useful to capture the possible contagion in the banking system, especially during the financial crisis when banks' defaults might be highly correlated.*

- *Estimates of asset correlations show that they are time-varying and positive. The multiple default risk index indicates that the most stressful period for the banking system was the Asian financial crisis. The results also show that the index jumped in advance of the crisis. The index with early-warning capability may serve as a vulnerability indicator for the banking sector in addition to the aggregate measure of individual banks' default probabilities.*

- *The multiple default risk index has declined steadily since 1999 due to the economic recovery in Hong Kong and consolidation in the banking sector.*

- *The study shows that, from a financial stability perspective, the multiple default risk index derived from the extended Merton model, together with individual banks' default probabilities, is useful for monitoring systemic risk in the banking system.*

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Non-Technical Executive Summary

The assessment of credit risk of the banking system has long been one of the focuses of central banks from a financial stability perspective. Given that defaults in banks are highly correlated, the monitoring tool should be able to capture the potential contagion effect among banks. Like other central banks, this study develops an indicator based on an assessment of the likelihood for at least one bank to default within a short time period to monitor the systemic risk of the banking sector. The study extends the Merton model and applies it with the probability theory to assess the one-year multiple default risk of a portfolio of publicly-listed banks in Hong Kong during the period of January 1997 to January 2006. The advantage of this approach is that, by incorporating asset correlations between banks which capture the possible spillover and contagion effects in the banking system, it provides an indicator for the risk of joint default in a closed and tractable form.

The analysis shows that the estimated asset correlations between banks are positive and time-varying. A high positive level of asset correlation between banks may contribute to systemic risk among banks during crises and would make the banking system more vulnerable. Based on the derived multiple default risk index, the most stressful period for the banking system was the Asian financial crisis. The results also show that the index jumped in advance of the crisis. The multiple default risk has declined steadily throughout the years since 1999. From mid-2004 onwards, it has fallen below the pre-Asian financial crisis level, indicating that the likelihood of multiple defaults in the banking system has eased substantially from the peak in the fall of 1998.

The multiple default risk index with early-warning capability is an informative measure of the systemic risk of the banking system. When used in conjunction with individual banks' default probabilities and market intelligence, this indicator can be considered as an effective monitoring tool to aid the ongoing surveillance work of regulators.

I. INTRODUCTION

Banks face different kinds of risks when conducting their business, such as interest rate risk, operational risk, payments and settlement risk, and credit risk. Among these risks, the assessment of credit risk of the banking system has long been one of the focuses of central banks from a financial stability perspective. For instance, since late 2004, both the Bank of England (BOE) and the European Central Bank (ECB) have published their estimated indicators based on the Merton-type model as part of the measures of banking system vulnerability in their regular *Financial Stability Review* (BOE (2004) and ECB (2004)).[1] The International Monetary Fund (IMF) also reports similar indicators in its *Global Financial Stability Report* (IMF (2004) and IMF (2005)).

In addition to an aggregate measure of individual banks' default probabilities, central banks also use an indicator based on an assessment of the likelihood for at least one bank to default within a short time period (rather than the probability of an individual default) in monitoring the systemic risk of the banking system. In this regard, the IMF (2005) modifies its credit risk indicators (CRIs) to reflect the probability of multiple defaults within a portfolio of banks.[2] By taking into account the correlation among a group of banks, the US Federal Reserve Board (FRB) simulates the probability of multiple defaults such that more than one large financial institution among a group of about 40 large financial institutions may default within a period of one year and use this as one of the financial indicators to monitor the risks in the US financial market regularly (Nelson and Perli (2005)).

The methodology adopted by Nelson and Perli (2005) requires the simulation of the market value of assets of each bank in the portfolio, which is computationally intensive. It is also hard to verify whether the simulated pattern of asset values is realistic. An alternative approach by Cathcart and El-Jahel (2004) calculates the probability of multiple defaults based on probability theory. The advantage of this approach is that it provides the joint probabilities for default of at least one bank in a closed and tractable form.

This study applies the Cathcart and El-Jahel's approach to estimate the multiple default risk (expressed as an index) of a portfolio of publicly-listed banks in Hong Kong.[3] By incorporating asset correlations between two banks, we are able to capture the possible spillover and contagion effects of a bank's default on the others in the risk assessment. The purpose of this analysis is to illustrate how we can extend the Merton model,

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[1] The structural approach proposed by Merton (1974) is a standard framework frequently used by market participants, as well as central banks and international organisations, to help assess the default risk of banks. See Yu and Fung (2005) for a review of the Merton model.

[2] The IMF (2004) develops the CRIs that measure the default probabilities associated with first-to-default swaps of a portfolio of banks. The idea is to examine how the market perceives the credit risk of a portfolio of banks through the swap spreads.

[3] The portfolio of publicly-listed banks includes holdings companies which have overseas assets and / or may engage in businesses other than banking.

originally designed for the estimations of individual banks' default probabilities, to the case of multiple defaults and how the results can be used to assess the systemic risk of the banking system for surveillance purposes.

The paper is organised as follows. Section II discusses the methodology of the extended Merton model, the implementation of the probability theory to the model and the data. Section III presents the results of the estimation and examines the behavior of the indicator. Section IV concludes.

II. METHODOLOGY AND DATA

(a) The Model

The structural approach proposed by Merton (1974) assumes that the default process is related to the capital structure of a company. When the value of the company's asset (V_A) is less than the book value of its debt (X), the company is considered to be in default. Technical details on how the default probability of a single company is computed based on the Merton model are given in the Appendix. Using market data of equity values and accounting data, the default probability at the end of a period of time t (PD_t) is, according to the Merton model, given by:

$$PD_t = \phi\left(-DD(X_t, V_A, \sigma_A^2)\right) \tag{1}$$

where ϕ is the standard cumulative normal distribution,

DD is the "default distance" and is equal to $\dfrac{\ln\dfrac{V_A}{X_t}+\left(\mu-\dfrac{\sigma_A^2}{2}\right)t}{\sigma_A\sqrt{t}}$, (2)

V_A is the current market value of the company's assets,

μ and σ_A are the drift rate and volatility of V_A respectively, and

X_t is the book value of its debt due at time t.

In a multiple company setting, the asset value (V_{Ai}) of an individual company follows a stochastic process with drift μ_i and volatility σ_{Ai}:

$$dV_{Ai} = \mu_i V_{Ai} dt + \sigma_{Ai} V_{Ai} dz_i \tag{3}$$

dz_i is a Wiener process. For any two companies i and j, $i = 1, \ldots, N$ and $j = 1, \ldots, N$, dz_i and dz_j are their respective Wiener processes such that $dz_i dz_j = \rho_{ij} dt$. ρ_{ij} is defined as the asset correlation of the two companies i and j. Given the estimated default probability of individual company (PD_{it}), and the asset correlation ρ_{ij}, we can apply the probability theory to calculate multiple default probabilities of a portfolio of two to N companies.

(b) Probability Theory

In the case of two companies ($N = 2$), the probability of joint default at the end of a period of time t, denoted as $PD(1 \cap 2)_t$, is given by:[4]

$$PD(1 \cap 2)_t = \phi_2\left\{-DD_1(X_{t1}, V_{A1}, \sigma_1^2), -DD_2(X_{t2}, V_{A2}, \sigma_2^2), \rho_{12}\right\} \tag{4}$$

where ϕ_2 is the bivariate standard cumulative normal distribution function, $DD_i(X_{ti}, V_{Ai}, \sigma_i^2)$ for $i = 1,2$ is the default distance of company i given by Equation (2), and ρ_{12} is the asset correlation between companies 1 and 2. The probability of at least one company may default ($PD(1 \cup 2)_t$) at the end of a period of time t is:[5]

$$PD(1 \cup 2)_t = PD_{t1} + PD_{t2} - PD(1 \cap 2)_t \tag{5}$$

When $N > 2$, the joint default probability will involve the multivariate cumulative normal distribution function and pairwise asset correlations between the companies and is given as:

$$PD(1 \cap 2 \cap \ldots \cap N)_t =$$

$$\phi_N\{-DD_1(X_{t1}, V_{A1}, \sigma_1^2), \ldots, -DD_N(X_{tN}, V_{AN}, \sigma_N^2), \rho_{12}, \ldots, \rho_{1N}, \rho_{23}, \ldots, \rho_{2N}, \ldots, \rho_{N-1N}\} \tag{6}$$

where ϕ_N is the N-th variate standard cumulative normal, DD_i and ρ_{ij} are specified as before.

For N companies, one can derive the probability of at least one default ($PD(1 \cup 2 \cup 3 \cup \ldots \cup N)$) as follows:[6]

$$PD(1 \cup 2 \cup 3 \cup \ldots \cup N) = S_1 - S_2 + S_3 - S_4 + \ldots \pm S_N \tag{7}$$

where

$S_1 = \sum PD(i)$ with $PD(i)$, $i = 1, \ldots, N$, as individual company's default probability,

$S_2 = \sum PD(i \cap j)$, $i \neq j$, with $PD(i \cap j)$, $i = 1, \ldots, N$, $j = 1, \ldots, N$, as the joint default

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[4] The symbol $\cap$ stands for "and".
[5] The symbol $\cup$ stands for "or".
[6] For the proof of this probability result, see Feller (1950).

probability of two companies,

$S_3 = \sum PD(i \cap j \cap k)$, $i \neq j \neq k$, with PD($i \cap j \cap k$), i = 1, …, N, j = 1, …, N and k = 1, …, N, as the joint default probability of three companies,

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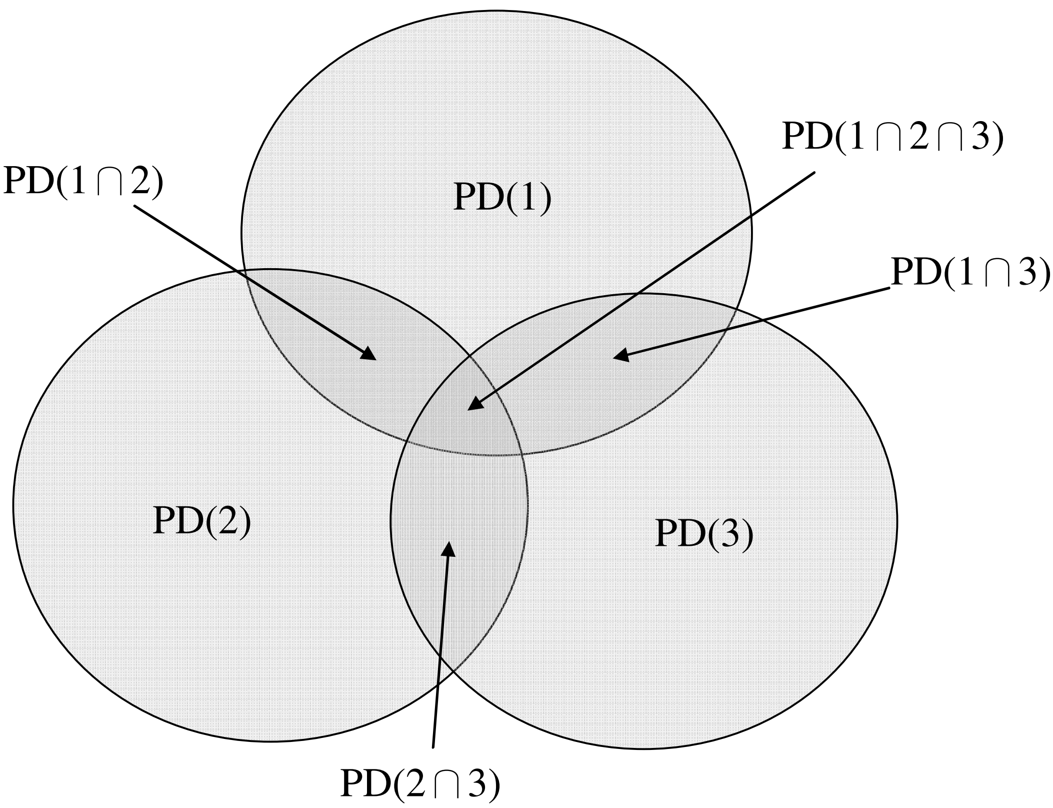
$S_N = PD(1 \cap 2 \cap 3 \cap 4 \cap \ldots \cap N)$, which is the joint probability of N defaults, and

each S_m has ${}_N C_m$ combination of terms, where ${}_N C_m = \dfrac{N!}{m!(N-m)!}$, which represents the number of different bivariate combination of size $m \leq N$.

As an illustration, when N = 3, the probability of at least one default, as given by Equation (7), is:

$$PD(1 \cup 2 \cup 3) = \{PD(1) + PD(2) + PD(3)\} - \{PD(1 \cap 2) + PD(1 \cap 3) + PD(2 \cap 3)\} + PD(1 \cap 2 \cap 3)$$

which can be represented graphically as:



In the following analysis, we apply Equations (6) and (7) to calculate the probability of at least one default in a portfolio of listed banks in Hong Kong. This probability is then expressed as a multiple default risk index to show the relative vulnerability of the banking system to a reference period.

(c) Asset Correlation

An essential component in the derivation of the joint default probability is the measure of asset correlation between companies (ρ_{ij}). There are various ways to estimate pairwise correlations, including sample correlation using rolling windows or correlations based on a factor model. In this study, we adopt the approach used by Nelson and Perli (2005) and Lehar (2005), which estimate the asset correlation with a simple exponentially-weighted moving average (EWMA) model.

For each company, we estimate its individual default probability and market value of asset (V_A) according to the model in Section II (a) above. To measure the correlations and volatilities of a portfolio of companies, for each month in the sample period we estimate a variance-covariance matrix of asset returns using the EWMA model with a decay factor λ of 0.94.[7] The variance of asset return is given by

$$\sigma_{i,t}^2 = (1-\lambda)R_{i,t}^2 + \lambda\sigma_{i,t-1}^2 \tag{8}$$

where $R_{i,t}$ is the monthly asset return of company *i*. The covariance $\sigma_{i\,j,t}^2$ between the asset returns of companies *i* and *j* is estimated by

$$\sigma_{i\,j,t}^2 = (1-\lambda)R_{i,t}R_{j,t} + \lambda\sigma_{i\,j,t-1}^2 \tag{9}$$

Once the variance-covariance matrix is estimated, the time-varying pairwise asset correlation can be determined as follows:

$$\rho_{i\,j,t} = \frac{\sigma_{i\,j,t}^2}{\sigma_{i,t}\sigma_{j,t}} \tag{10}$$

[7] We recognise that there are others ways to estimate volatility. Nevertheless, we use the EWMA model in this study as it is a standard model for market risk management. A comparison of different volatility estimation methods can be found in J.P. Morgan and Co. (1995). In the study, it is shown that a decay factor of 0.94 in the EWMA model gives the best forecast of volatility as compared to other methods such as GARCH-type models.

(d) Data

Next we illustrate the use of the model, using data of a portfolio of the publicly-listed banks in Hong Kong from January 1997 to January 2006.[8] For each bank, its one-year PD is estimated on a monthly basis. The data needed for the estimation include the market value of equity (V_E), the volatility of equity (σ_E), the debt level (X) and the one-year risk-free interest rate (r). The market value of equity is equal to the product of the outstanding number of shares and stock price.[9] The volatility of equity prices is estimated by the EWMA method.[10]

To calculate the debt level of a bank, "short-term loans", "due to creditors", "long-term loans" and "other long-term liabilities" in the balance sheet are considered.[11] The face value of debt is equal to (short-term loans + due to creditors) plus half of (long-term loans + other long-term liabilities).[12] As audited balance sheet data are available on an annual basis, monthly figures are estimated by using a cubic interpolation routine.[13] For simplicity, the expected drift rate of asset is assumed to be zero, given the short estimation horizon. Once the PDs of individual banks and ρ_{ij} of pairwise asset correlations are estimated, the probability of at least one default in the portfolio of banks can be established using Equations (6) and (7).

III. EMPIRICAL RESULTS

Chart 1 presents the median one-year PDs of the listed banks as well as their first and third quartiles over the sample period. Chart 2 shows the aggregate one-year PD weighted by the market capitalisations of the listed banks in the portfolio.

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[8] The banks in the portfolio represent about 80% of assets in the locally-incorporated authorized institutions as of the end of 2004. The number of institutions in this study ranges from 10 to 12 during the study period.

[9] Outstanding number of shares and stock prices are obtained from Bloomberg and Datastream.

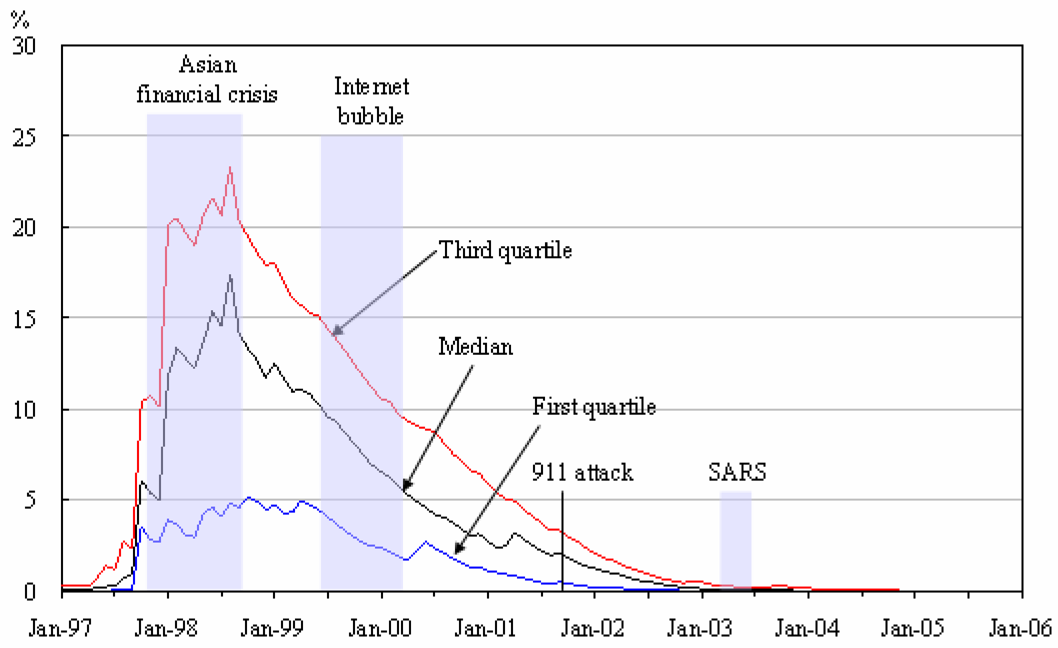
[10] Equity volatility is given by $\sigma_t^2 = (1-\lambda)R_t^2 + \lambda\sigma_{t-1}^2$, where R_t is the monthly return of equity price and λ is the decay factor which is set to be 0.94. The initial σ_t^2 is estimated from the average of the first 12 observations of the data series.

[11] The data are from Bloomberg.

[12] The reason to use short-term loans and due to creditors is that debt due within one year is more likely to cause default. The reasons for including long-term liabilities in the calculation are two-fold, (i) companies need to service their long-term debt, and these interest payments are part of their short-term liabilities; and (ii) the size of the long-term debt may affect the ability of a company to roll over its short-term debt. Similar to other studies, a factor of 0.5 is used for the long-term debt because the default point is found to lie generally somewhere between total liabilities and short-term liabilities (Crosbie and Bohn, 2002).

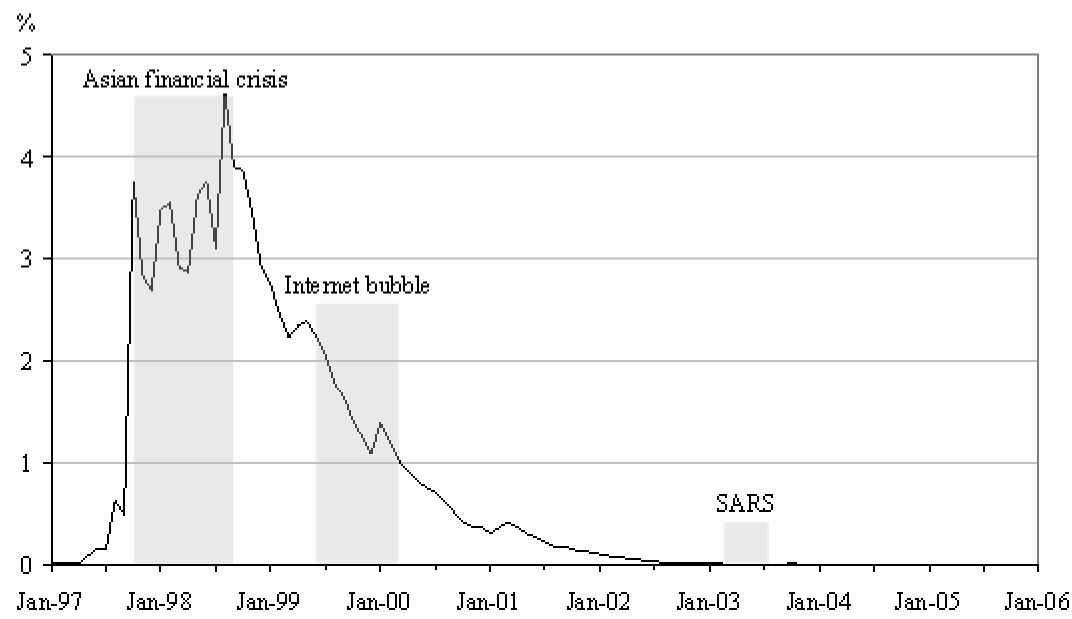
[13] The same cubic smoothing method is also applied to extrapolate the debt levels of a bank for recent months when the current balance sheet data are not available.

Chart 1. One-year Default Probability of Listed Banks



Note: Shaded areas represent major events or crises.
Source: HKMA staff estimates.

Chart 2. Aggregate One-year Default Probability of Listed Banks

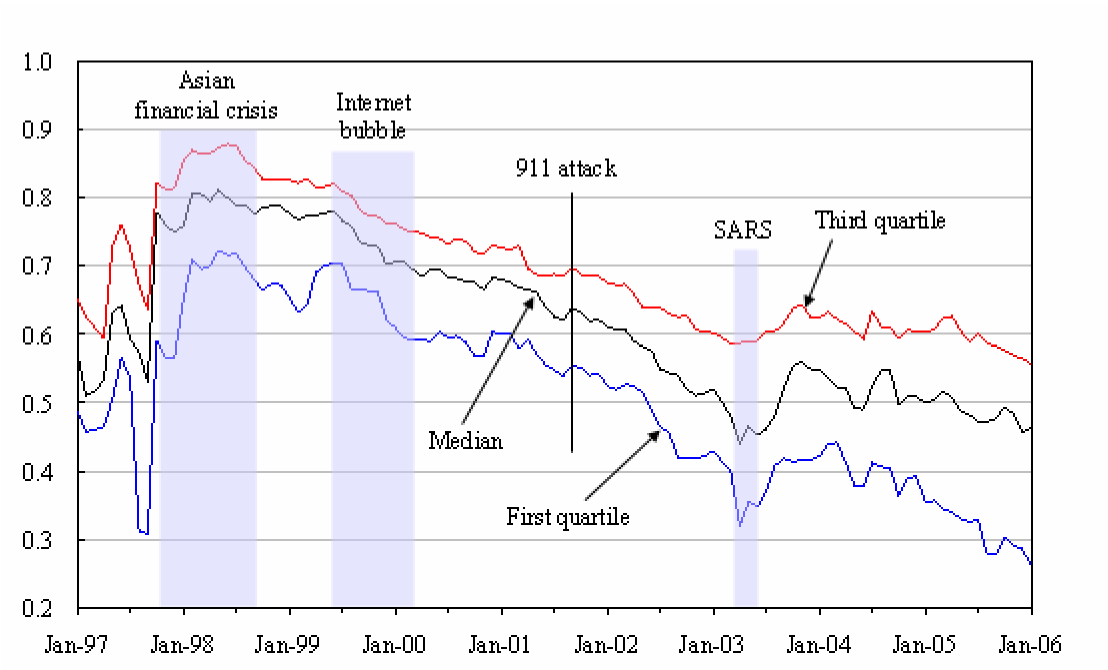


Note: Shaded areas represent major events or crises.
Source: HKMA staff estimates.

It is shown in Chart 1 that the median PDs rose sharply during the Asian financial crisis period, from 1% in September 1997 to as high as 17% in August 1998. The PDs of the banks in the third quartile reached over 20% in August 1998, signaling high vulnerability among this group of the banks, while those under the first quartile were about 5%. Interestingly, the PDs were not affected in other events. The aggregate one-year PD in Chart 2 depicts a similar trend. The biggest threat to the banking system was during the Asian financial crisis period when the aggregate PD started to climb in October 1997 and rose to 4.6% in August 1998. The aggregate PD declined steadily after the Asian financial crisis, and it was not affected by other events over the rest of the study period.

It is however noted that the aggregate PD may be distorted by one or two very large banks in the sample. In order to complement the aggregate PD for a better measure of the systemic risk in the banking system as a whole, and to capture the contagion effect among the banks in the banking sector, the probability of multiple defaults within a period of time is estimated and this probability is expressed as a multiple default risk index to show the relative vulnerability of the banking system to a reference period. To do so, we first need to derive the asset correlations between banks. Chart 3 shows the time-varying asset correlations between banks (as calculated by Equation (10)) over the sample period.

Chart 3. Monthly Asset Correlation between Banks

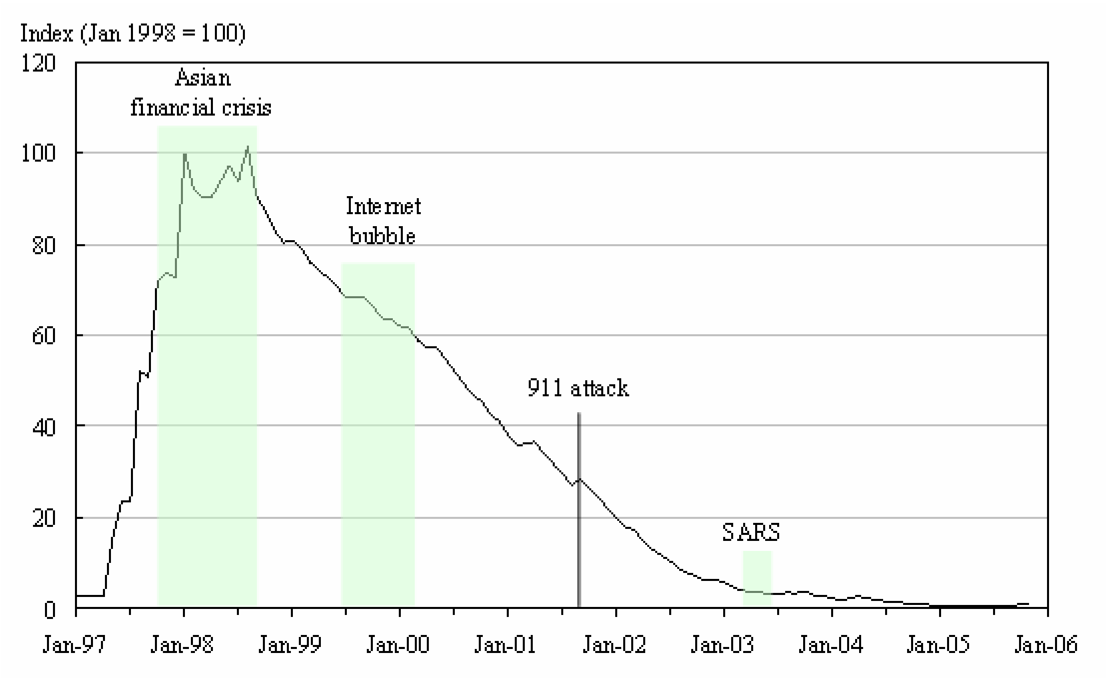


Note: Shaded areas represent major events or crises.
Source: HKMA staff estimates.

The estimated asset correlations in Chart 3 are positive and time-varying. The notable sharp increase in the correlations occurred during the Asian financial crisis, when the median correlation jumped from 0.53 in September 1997 to 0.78 in October 1997. The correlations among these banks remained at such a high level over the crisis period, with the first quartile around 0.70 level and the third quartile over 0.85. Such high level of asset correlation may contribute to systemic risk among banks during crises and would increase the vulnerability of the banking system. The correlations declined after the Asian financial crisis but picked up again after the SARS episode. Since mid-2003, the asset correlation has resumed its declining trend.

Based on the default probabilities of individual banks and their pairwise asset correlations, we can use the results from Equations (6) and (7), which estimate the probability of at least one default within the portfolio of listed banks, to derive the multiple default risk index for the banking system. Chart 4 presents the multiple default risk index for the banking system with the base period in January 1998.

Chart 4. Multiple Default Risk Index
(based on One-year Probability of At Least One Default of Listed Banks)



Note: Shaded areas represent major events or crises.
Source: HKMA staff estimates.

Similar to the aggregate one-year PD shown in Chart 2, the multiple default risk index indicates that the most stressful period for the banking system was the Asian financial crisis from October 1997 to September 1998. Given the severity of the financial crisis in 1997-98, the multiple default risk index can be used in addition to the aggregate PD in Chart 2 for the monitoring of systemic risk and trend in the banking sector. Furthermore, it is shown that the multiple default risk index jumped in advance of the financial crisis. It reached a reading of 23.7 in June 1997, from just 2.6 in April 1997, and continued to climb throughout the year. The early-warning capability of this index on the systemic risk of the banking system is as informative as the aggregate PD.[14]

Systemic risk in the banking system remained high after the Asian financial crisis, but it has declined steadily throughout the years since 1999. The risk of multiple defaults during the internet bubble period did not stand out as prominently as in the US.[15] The threat of the global liquidity and settlement crisis during and after the 911 attack in the US was not perceived as a substantial threat to the banking system in Hong Kong. The outbreak of the SARS epidemic in 2003 had little impact on the vulnerability of the banking system too. From mid-2004 onwards, the index has fallen below a reading of 2, indicating that the risk of multiple defaults in the banking system as compared to the period in January 1998 has eased substantially. It takes over five years for the index to return back to its pre-Asian financial crisis level, partly due to the prolonged economic recession in Hong Kong following the Asian financial crisis and the "negative equity" problem in some banks' loan portfolios.[16],[17]

As suggested by Nelson and Perli (2005), it would be more informative to compare the relative levels of default indicator at different points in time. For instance, while the multiple default risk index was estimated at a level of 28 in September 2001 (the 911 attack), it is also of interest to note that the risk of at least one bank may default within one year was over three times less than that during the Asian financial crisis.

Mainly due to the scarcity of data caused by the lack of actual default events in Hong Kong, the accuracy of PDs of individual banks estimated in this study has not been tested empirically. Nonetheless, these PDs, in association with asset correlations, help to derive the multiple default risk index which is a useful measure of the systemic vulnerability of the banking system, in addition to the aggregate PDs.

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[14] Before the aggregate PD surged to about 3.8% in October 1997 when the Asian financial crisis first broke out, it remained at a very subdue level of 0.6% or below. Policymakers focusing only on this measure would find it difficult to extract any early warning to the potential systemic risk of the banking system.

[15] In the US, the probabilities of multiple defaults in the spring of 2000 were as high as that in the fall of 1998.

[16] Compared to the simulated indicator derived by the FRB (Nelson and Perli (2005)) with about 40 large financial institutions, the decline in the probability of at least one default of financial institutions in the US during the post-crisis period is more rapid.

[17] Factors that may contribute to the rise and fall of the aggregate PDs and the multiple default risk index will be examined in a separated research paper.

IV. CONCLUSION

The assessment of vulnerability of the banking system is an important challenge to regulators responsible for banking and financial stability. One of the credit risk models that has been widely used by central banks is the Merton model. This paper extends the Merton model and applies it to assess the multiple default risk of a portfolio of publicly-listed banks in Hong Kong during the period of January 1997 to January 2006. In our estimation, the derived multiple default risk index indicates that the most stressful period of the banking system was during the Asian financial crisis. The results also show that the index jumped in advance of the crisis. Since 1999, the systemic risk of the banks has declined steadily due to the economic recovery in Hong Kong and consolidation in the banking sector.

The extended Merton model, by incorporating asset correlations between banks, has the advantage of providing high frequency information that can be used to assess the systemic risk in the banking system. This study has shown that, from a financial stability perspective, the multiple default risk index with early-warning capability is as informative a measure of the systemic risk of the banking system as the weighted aggregate default probability. The multiple default risk index derived from this approach, when used together with aggregate and individual banks' default probabilities as well as market intelligence, can be considered as an effective monitoring tool to aid the ongoing surveillance work of regulators.

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Appendix

Technical Details for Deriving the Default Probability Based on the Structural Approach

To derive the default probability using the Merton approach, the market value of the company's underlying assets is assumed to follow the stochastic process:

$$dV_A = \mu V_A dt + \sigma_A V_A dz \tag{A1}$$

where V_A , dV_A are the company's asset value and change in asset value,
μ , σ_A are the company's asset value drift rate and volatility, and
dz is a Wiener process.

The probability of a company default is given by the likelihood that the market value of the company's assets will be less than the book value of the company's liabilities by the time when the debt matures, that is:

$$\text{PD}_t = \text{Prob}[V_A^t \le X_t \big| V_A^0 = V_A] = \text{Prob}[\ln V_A^t \le \ln X_t \big| V_A^0 = V_A] \tag{A2}$$

where PD_t is the probability of default by time t, V_A^t is the market value of assets at time t, and X_t is the book value of debt due at time t.

Given (A1), the value of the company's assets at time t, V_A^t, is:

$$\ln V_A^t = \ln V_A + \left(\mu - \frac{\sigma_A^2}{2}\right)t + \sigma_A \sqrt{t}\varepsilon \tag{A3}$$

where V_A the company's current asset value,
μ is the expected return on the company's assets, and
ε is the random component of the company's return.

Combining (A2) and (A3) gives the PD as

$$\text{PD}_t = \text{Prob}\left[\ln V_A + \left(\mu - \frac{\sigma_A^2}{2}\right)t + \sigma_A \sqrt{t}\,\varepsilon \le \ln X_t\right]$$

or, alternatively

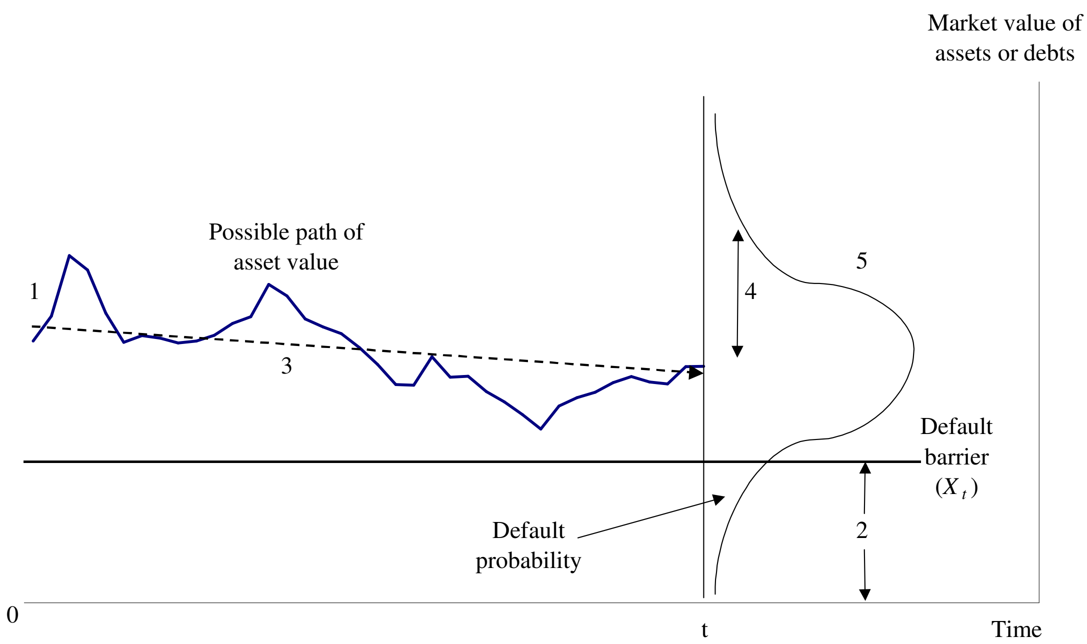
$$\mathrm{PD_t} = \mathrm{Prob}\left[-\frac{\ln\frac{V_A}{X_t} + \left(\mu - \frac{\sigma_A^2}{2}\right)t}{\sigma_A\sqrt{t}} \geq \varepsilon \right] \quad \text{(A4)}$$

The Black-Scholes model assumes that the random component of the asset return is normally distributed, $\varepsilon \sim N(0,1)$ and as a result the probability of default can be defined in terms of the cumulative normal distribution as:

$$\mathrm{PD_t} = N\left[-\frac{\ln\frac{V_A}{X_t} + \left(\mu - \frac{\sigma_A^2}{2}\right)t}{\sigma_A\sqrt{t}} \right] \quad \text{(A5)}$$

See Chart A1 for an illustration.

Chart A1. Asset Value, Default Barrier and Default Probability



Notes:
1. The current market value of assets ($V_{A,}$).
2. The level of default barrier, the book value of debts due at time t (X_t).
3. The expected rate of growth in the asset value (μ).
4. The volatility of asset (σ_A).
5. The distribution of asset value (N(0.1)).

There are two unknowns, V_A and σ_A, in (A5) for the estimation of the default probability. These can be obtained by simultaneously solving equations (A6) and (A7).

$$V_E = V_A N(d1) - e^{-rt} X_t N(d2) \tag{A6}$$

where r is the risk free interest rate,

$$d1 = \frac{\ln(V_A / X_t) + (r + \sigma_A^2 / 2)t}{\sigma_A \sqrt{t}},$$

$$d2 = d1 - \sigma_A \sqrt{t},$$

$N(.)$ is the standard cumulative normal distribution.

and

$$\sigma_E = \eta_{E,A} \sigma_A = (V_A / V_E)\, \Delta\, \sigma_A \tag{A7}$$

where $\eta_{E,A} = (V_A / V_E)(\partial V_E / \partial V_A)$ is the elasticity of equity value to asset value,[18]

σ_E is the volatility of the company's equity value, and

Δ is the hedge ratio, $N(d1)$, from equation (A6).

Equation (A6) is the Black-Scholes pricing formula, which relates the market value of equity (V_E) to the market value and volatility of the company's underlying assets (V_A and σ_A), given that the company's capital structure is only composed of equity and debt, and given X_t the book value of the debt which is due at time t.

Equation (A7) links the volatility of equity value with that of the company's assets value which is assumed to follow the stochastic process shown in equation (A1).

[18] See Bensoussan et al. (1994).