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*CURRENCY BARRIER OPTION PRICING WITH MEAN REVERSION**

Key points:

- *Currency option traders usually use the Black-Scholes model in which the exchange rate follows a lognormal process. However, it is found that exchange rates may follow a mean-reverting process instead, for example, certain currencies are constrained to move inside target zones or under a managed-floating regime. Different dynamical processes of exchange rates raise uncertainty on the choice of a pricing model for currency options. Such model risk would worsen the market condition when there is an adverse shock on the underlying currency. Financial instability could thus result, if pricing models are not chosen and used properly in the foreign exchange market.*

- *Barrier options have emerged as significant products for hedging and investment in the foreign exchange market since the late 1980s, largely in the over-the-counter markets and for structuring financial products (e.g., currency-linked notes). The existence of a barrier option depends upon whether the underlying exchange rate has crossed a predetermined barrier prior to the exercise time. The estimated daily turnover of currency barrier option trading is about US$12 billion.*

- *This paper develops a barrier-option pricing model in which the exchange rate follows a mean-reverting lognormal process. The corresponding closed-form solutions for the barrier options with time-dependent barriers are derived.*

- *The mean-reverting lognormal process keeps the exchange rate in a range around the mean level. The numerical results show that the parameters of the mean-reverting lognormal process make the valuation of currency barrier options and their hedge parameters different from those obtained from the conventional Black-Scholes model.*

- *A non-technical summary of the paper is in the following section.*

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NON-TECHNICAL SUMMARY

Currency option traders use the Black-Scholes model in which the exchange rate follows a lognormal process. However, it is found that the exchange rate may follow a mean-reverting process instead because of the following reasons:

(i) In the long run the equilibrium exchange rate is determined by domestic money stock, interest rate level, and national income relative to the same quantities in the foreign economy. However, in the short run it may deviate (or overshoot) from equilibrium, due to the sluggishness with which goods prices react to disturbances;

(ii) Mean reversion could come about due to interventions in the foreign exchange market by the central banks; and

(iii) Certain currencies are constrained to move inside target zones or under a managed-floating regime. We might expect mean reversion of the exchange rate when the central banks engage in intramarginal intervention and market participants expect the exchange rate band to be fully credible and engage in stabilising speculation.

Different dynamical processes of exchange rates raise uncertainty on the choice of a pricing model for currency options.

Option traders usually adopt dynamic hedging based on hedging parameters from option models to manage their option portfolios.[1] If these hedging-related transactions are large relative to the underlying market, the hedging strategy could make significant demands on market liquidity and lead to higher market volatility. As traders in general use Black-Scholes model for their currency option portfolios, they may respond to a perceived increase in the riskiness or even losses of their positions by paring back the size of those positions. This may have the collective and unintended consequence of reducing market liquidity at the time when it is most needed and can destabilise markets.[2] Clearly, uncertainty of the choice of an option model can lead to "model risk" that would worsen the market condition when there is an adverse shock on underlying asset values of options. Therefore, the benefits of derivatives including options, both to individual institutions and to the financial system and the economy as a whole, could be diminished, and financial instability could result, if pricing models are not chosen and used properly.

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[1] Dynamic hedging eliminates the risk of the option position by trading continuously the underlying asset of the option.

[2] See R. W. Ferguson (2002)'s speech on "Financial Engineering and Financial Stability" about dynamic hedging and herding behaviour.

Barrier options have emerged as significant products for hedging and investment in the foreign exchange market since the late 1980s, largely in the over-the-counter markets and for structuring financial products (e.g., currency-linked notes). The estimated daily turnover of currency barrier option trading is about US$12 billion.[3] Barrier options are path-dependent options which are usually structured as modifications of simple European puts and calls. The existence of a European barrier option depends upon whether the underlying asset price has crossed a predetermined barrier prior to the exercise time. For example, an up-and-out call pays off the usual European-call payoff at expiry, unless at any time before expiry the underlying asset has been traded at or higher than the barrier level.

This paper develops a barrier-option pricing model in which the exchange rate follows a mean-reverting lognormal process. The corresponding closed-form solutions for the barrier options with time-dependent barriers are derived. The numerical results show that barrier option values and the corresponding hedge parameters under the proposed model are different from those based on the Black-Scholes model. For an up-and-out call, the mean-reverting process keeps the exchange rate in a small range around the mean level. When the mean level is below the barrier but above the strike price, the risk of the call to be knocked out is reduced and its option value is enhanced compared with the value under the Black-Scholes model. The parameters of the mean-reverting lognormal process therefore have a material impact on the valuation of currency barrier options and their hedge parameters.

[3] According to BIS Triennial Central Bank Survey of Foreign Exchange and Derivatives Market Activity in 2004, the daily average turnover of option transactions in all currencies was US$117 billion. D. Luenberger and R. Luenberger estimate that barrier option trading accounts for 10% of all traded options (see Pricing and hedging barrier options. Investment practice, Stanford University, EES-OR, Spring 1999).

1. INTRODUCTION

Barrier options are path-dependent options that are usually structured as modifications of simple European puts and calls. The existence of a European barrier option depends upon whether the underlying asset price has crossed a predetermined barrier prior to the exercise time. For example, an up-and-out call pays off the usual European-call payoff at expiry, unless at any time before expiry the underlying asset has been traded at or higher than the barrier level. In this example, it is said to *knock-out*, becoming worthless. On the other side of the picture are "in" options, which actually commence when the underlying price touches the barrier. Barrier options are cheaper than regular European options because of these extinguishing and activating features. Thus they are attractive to investors who are averse to paying high premiums. In addition, sellers of barrier options may be able to limit their downside risk. Barrier options have emerged as significant products for hedging and investment in foreign exchange, equity and commodity markets since the late 1980s, largely in the over-the-counter markets and for structuring financial products (e.g., structured notes linked to exchange rates). The valuation of barrier options has been well covered in literature (see Hui, 1996, 1997; Kunitomo & Ikeda, 1992; Merton, 1973, Rich, 1994; Rubinstein & Reiner, 1991). The dynamics of the underlying asset price in the valuation follow the lognormal process proposed by Black and Scholes (1973).

Regarding currency options, Garman and Kohlhagen (1983) adapt the Black-Scholes model to develop the currency option valuation model. However, it is not entirely satisfactory because the ordinary Black-Scholes model is for stock options and currencies are different from stocks in important respects. In this connection, Sørensen (1997) and Ekvall, Jennergren, and Näslund (1997) present revised currency option pricing models in which the exchange rate follows a mean-reverting process (i.e., the logarithm of the exchange rate follows an Ornstein-Uhlenbeck process). Sørensen (1997) proposes an equilibrium model that establishes a mean-reverting process for the exchange-rate dynamics through its effect on the dynamics in the domestic and foreign term structures of interest rates. Ekvall et al. (1997) state several reasons why a mean-reverting process may be reasonable for exchange rates, even though these three reasons for mean reversion are not all in effect simultaneously. First, in the long run the equilibrium exchange rate is determined by domestic money stock, interest-rate level, and national income relative to the same quantities in the foreign economy. However, in the short run it may deviate (or overshoot) from equilibrium, because of the sluggishness with which goods prices react to disturbances. The resulting tendency of the exchange rate to return to an equilibrium can be thought of as mean reversion. Such a sticky-price monetary approach incorporating overshooting has been discussed by Dornbusch (1976).

Second, mean reversion could come about because of interventions in the foreign-exchange market by the central banks. Such interventions have been shown in many studies (see MacDonald, 1988). An underlying feature of the interventions is that the authorities try to bring the exchange rate back to some normal or equilibrium level. A natural way to model this is through a mean-reverting process for the exchange rate.

Third, certain currencies are constrained to move inside target zones or under a managed-floating regime.[4] An important prediction of the theoretical literature on targeted exchange rates is that we might expect mean reversion of the exchange rate when the central banks engage in intramarginal intervention, and market participants expect the exchange-rate band to be fully credible and engage in stabilising speculation. This mean-reverting property is widely referred to in the literature (see, for example, Anthony & MacDonald, 1998; Krugman, 1991; Rose & Svensson, 1994; Svensson, 1992, 1993). Several recent studies have attempted to investigate this theoretical prediction empirically by examining the time-series properties of the currencies participating in the European monetary system (see, for example, Anthony & MacDonald, 1998, 1999; Ball & Roma, 1993, 1994; Kanas, 1998; Nieuwland, Verschoor, & Wolff, 1994; Rose & Svensson, 1994; Svensson, 1993). Although their investigations had mixed results, the empirical results suggest that mean reversion is present.

Sørensen (1997) finds that the mean-reverting process has significant implications for the valuation of American currency options and optimal exercise strategies. As barrier options are path-dependent options like American options, implications of mean reversion for the valuation of currency barrier options could also be significant. This paper develops a barrier option pricing model in which the exchange rate follows a mean-reverting lognormal (MRL) process to study the implications. The corresponding closed-form solutions for the barrier options with time-dependent barriers are derived. The pricing solutions are used to examine the effects of the MRL dynamics on the values and hedge parameters of European-style currency barrier options. The results are compared with those generated from the Black-Scholes model. As mean reversion could be present in foreign exchange dynamics and the use of currency barrier options for hedging foreign exchange exposures and structuring financial products is popular in the financial market, the findings in this paper could provide an analytical valuation framework for research in currency option pricing.

The model developed here may also be applicable for pricing barrier options on commodity products or other assets that follow a mean-reverting process.

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[4] Examples are the currencies within the European Monetary System in 1990s, the Chinese renminbi, Hong Kong dollar, Singapore dollar and Malaysia ringgit.

The paper is organized as follows. The next section presents the currency barrier option pricing model under the MRL process and derives the corresponding option pricing formulas. Next the effects of the MRL dynamics on the value of an up-and-out call and its hedge parameters are examined. The results are compared with those under the Black-Scholes model and those of a regular call under the MRL process. The final section summarises the findings.

2. PRICING EUROPEAN BARRIER OPTIONS

In the MRL model, it is assumed that the exchange rate *F* (i.e., the domestic currency value of a unit of foreign currency) evolves according to the diffusion process specified as

$$dF = \left[\kappa(\ln F_0 - \ln F) + \mu\right]Fdt + \sigma F dW \tag{1}$$

where F_0 is the conditional mean exchange rate[5], κ is the parameter measuring the speed of reversion to this mean, σ is the volatility of the exchange rate, μ is the instantaneous return on *F*, and W is a standard Wiener process so that dW is normally distributed.

It is assumed that option prices depend on *F* as the only state variable. By the usual arbitrage-free argument for currency options, the risk-adjusted expected excess returns of holding a currency option and holding its underlying currency must be identical (see, for example, Garman and Kohlhagen, 1983). Applying Ito's lemma, the partial-differential equation governing the option price *P*(*F*, *t*) with time-to-maturity of *t* based on the model is

$$\frac{\partial P(F,t)}{\partial t} = \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 P(F,t)}{\partial F^2} + \left[\kappa(\ln F_0 - \ln F) + (r - r^*)\right]F\frac{\partial P(F,t)}{\partial F} - rP(F,t), \tag{2}$$

where *r* and *r** are the risk-free interest rates of the domestic and foreign currencies, respectively. By putting $x = \ln(F/H)$ and $\theta = \left[\ln(F_0/H) - \sigma^2/2\kappa + (r - r^*)/\kappa\right]$ where *H* is a level to determine the barrier, Equation (1) becomes

$$\frac{\partial P(x,t)}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 P(x,t)}{\partial x^2} + \kappa(\theta - x)\frac{\partial P(x,t)}{\partial x} - rP(x,t). \tag{3}$$

With further changing variables:

$$e^{-rt}\tilde{P}\left(xe^{-\kappa t}, t\right) = P(x,t), \tag{4}$$

$\tilde{P}(x,t)$ satisfies the following equation:

$$\frac{\partial \tilde{P}(x,t)}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 \tilde{P}(x,t)}{\partial x^2} + \kappa\theta\frac{\partial \tilde{P}(x,t)}{\partial x}. \tag{5}$$

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[5] F_0 can be interpreted as the historical mean instantaneous exchange rate.

Sørensen (1997) and Ekvall et al. (1997) derive regular call and put pricing formulas in which the exchange rate follows a mean-reverting process. The respective formulas are presented in Equations (A8) and (A14) in Appendix A. The solution of Equation (5) with an up-and-out[6] option-payoff condition of $\tilde{P}(x,t=0)$ at option maturity is

$$\tilde{P}(x,t) = \int_{-\infty}^{0} dx' \left[G(x,t;x',t=0) - G(x,t;-x',t=0)e^{-\beta x'}\right]\tilde{P}(x',t=0), \tag{6}$$

and the distribution function $G(x,t;x',0)$ is given by

$$G(x,t;x',t=0) = \frac{1}{\sqrt{4\pi c_1(t)}} \exp\left\{-\frac{\left[x'-x-c_2(t)\right]^2}{4c_1(t)}\right\}, \tag{7}$$

where

$$c_1 = \frac{\sigma^2}{4\kappa}\left(1-e^{-2\kappa t}\right), \tag{8}$$

$$\begin{aligned} c_2 &= \theta\left(1-e^{-\kappa t}\right) \\ &= \left[\ln(F_0/H) - \sigma^2/2\kappa + (r-r^*)/\kappa\right]\left(1-e^{-\kappa t}\right), \end{aligned} \tag{9}$$

and β is a real number. The payoff condition of an up-and-out call is $\max(F-K,0)$ if $F < h(t)$ and 0 if $F \geq h(t)$, where $h(t)$ is the moving barrier and *K* is the strike price at option maturity, such that

$$\begin{aligned} &\tilde{P}_{call}(x,t=0) = \max\left(He^{x}-K,0\right), \quad F < h(t), \\ &\tilde{P}_{call}(x,t=0) = 0, \quad F \geq h(t). \end{aligned} \tag{10}$$

Although closed-form solutions of fixed barrier options can be obtained under the Black-Scholes model, closed-form solutions of barrier options under the MRL model are obtained when the barrier is time dependent (i.e., moving with time).[7] The trajectory of the moving barrier $\tilde{h}(t)$ under the MRL model is

$$\tilde{h}(t) = -c_2(t) - \beta c_1(t),$$

or

$$h(t) = H\exp\left\{-e^{\kappa t}\left[c_2(t)+\beta c_1(t)\right]\right\}. \tag{11}$$

The option vanishes when

$$\tilde{P}_{call}\left(x=\tilde{h}(t),t\right) = 0 \tag{12}$$

The solution of a call is obtained by solving Equation (6) subject to the payoff condition (10) and the boundary condition (12). After substituting back the variables, if $h(t)$ is greater than the strike price *K*, the up-and-out call value is

6 It is a regular option that ceases to exist if the underlying asset price reaches a certain level, the barrier. The barrier level is above the initial asset price.

7 When the model parameters of barrier options are time dependent under the Black-Scholes model, there are no closed-form solutions (see Robert & Shortland, 1997 and Lo, Lee & Hui, 2003).

$$\begin{aligned} P_{uo-call}(F,t) &= \tilde{F}e^{-rt}N\left(a+\sqrt{2c_1}\right)-Ke^{-rt}N(a) \\ &- \tilde{F}e^{-rt}N\left(b+\sqrt{2c_1}\right)+Ke^{-rt}N(b) \\ &+ \tilde{F}\left(\frac{H}{\tilde{F}}\right)^{2-\beta} e^{\left[(\beta-1)^2-(\beta-1)\right]c_1-rt}\left[N\left(c+\sqrt{2c_1}\right)-N\left(d+\sqrt{2c_1}\right)\right], \\ &- K\left(\frac{H}{\tilde{F}}\right)^{-\beta} e^{\beta(\beta-1)c_1-rt}\left[N(c)-N(d)\right] \end{aligned} \tag{13}$$

where

$$\tilde{F} = \left(\frac{F}{H}\right)^{\exp(-\kappa t)-1} e^{c_1+c_2}F \tag{14}$$

$$a = \frac{\ln\left(\tilde{F}/K\right)-c_1}{\sqrt{2c_1}}. \tag{15}$$

$$b = \frac{\ln\left(\tilde{F}/H\right)-c_1}{\sqrt{2c_1}}, \tag{16}$$

$$c = \frac{\ln\left(H/\tilde{F}\right)-c_1-2(\beta-1)c_1}{\sqrt{2c_1}} \tag{17}$$

$$d = \frac{\ln\left(H^2/\tilde{F}K\right)-c_1-2(\beta-1)c_1}{\sqrt{2c_1}}, \tag{18}$$

and $N(.)$ is the cumulative normal distribution function.

To obtain the value of a fixed barrier (i.e., time-independent constant barrier) knock-out call, the parameter β can be adjusted such that the solution in Equation (13) provides the best approximation to the exact value of the fixed barrier call by using a simple method developed by Lo et al. (2003) for solving barrier option values with time-dependent model parameters. The method is based upon simulating the fixed barrier as a slowly fluctuating barrier with a small oscillating amplitude by tuning the parameter β. The upper and lower bounds (in closed form) provided by the method are also very tight for the exact fixed barrier option prices. Because the bounds and estimates of the option price appear in closed form, they can be computed very efficiently. Furthermore, the bounds can be improved systematically, and these improved bounds are again expressed (in closed form) in terms of the multivariate normal distribution functions.[8] In the next section, Figure 1 illustrates the movements of the time-dependent barriers with different κ over option life.

[8] Rapisarda (2004) applied the results of Lo et al. (2003) to derive in an analytical fashion the approximate prices of various types of barrier options, for example forward start/early expiry barriers and window barriers.

The derivations of option pricing formulas of a down-and-out call, up-and-out and down-and-out puts, and knock-in options are similar to that of the up-and-out call. They are presented in Appendix A.[9]

The resemblance between the barrier option pricing formulas derived above and the corresponding formulas with the exchange rate under the Black-Scholes model is obvious.[10] The random variable $\tilde{F}$ specified in Equation (14) above is lognormal, resulting in similar formulas as those under the Black-Scholes model. A $\tilde{\sigma}^2$, which is the variance of the logarithm of the price $\tilde{F}$, replaces σ^2 of the Black-Scholes model, which has the same meaning. In other words, $\tilde{F}$ is lognormal with $\mathrm{var}\left[\ln \tilde{F}\right] \equiv \tilde{\sigma}^2 / t$ where

$$\tilde{\sigma} = \sqrt{2c_1 / t} = \frac{\sigma}{\sqrt{2\kappa t}}\sqrt{1 - e^{-2\kappa t}} \ . \tag{19}$$

The resemblance is equivalent to the change of numeraire of F by $\tilde{F}$. Different magnitudes of the speed κ of reversion to this mean give some intuitive interpretations of $\tilde{F}$. When the speed is very strong (i.e., $\kappa >> 1$), $\tilde{F}$ converges to F_0. This means that the dynamical process of the exchange rate is almost deterministic, such that it will stick to F_0 with the effective volatility $\tilde{\sigma} \to 0$. Given an up-and-out call, its value converges to $\max\left[\left(F_0 - K\right), 0\right]e^{-rt}$ if F_0 is below the barrier $h(t)$ or 0 otherwise (i.e., the call is knocked out). This illustrates that the presence of mean reversion makes the associated barrier option values very different from those in the Black-Scholes model. The differences will be shown numerically in the following section. Conversely, when the speed is very weak (i.e., $\kappa \to 0$), the dynamical process specified in the model converges to a lognormal process where $\tilde{F} \cong F$ and $\tilde{\sigma} \cong \sigma$ such that the corresponding barrier option values are those based on the Black-Scholes model. Equation (19) and the limits of $\tilde{\sigma}$ with different κ show that $\tilde{\sigma}^2$ is a decreasing function with κ and is less than σ^2. The proof of $\tilde{\sigma}^2$ being a decreasing function with κ is given in Appendix B.

3. Effects of MRL Process on Barrier Option Values and Hedge Parameters

When dealers trade any derivative instruments, they often need to hedge their positions. It is important to determine the hedge parameters of the derivative instruments. The following discussion is focused on the effects of the mean-reverting process on the

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9 Based upon the method of multiple images proposed by Lo et al. (2006) for valuation of double-barrier options with time-dependent parameters, a similar solution in Equation (6) for double-barrier options with mean reversion can be obtained.

10 The derivation of the formulas under the Black-Scholes model can be found in Rubinstein and Reiner (1991) and Rich (1994).

values and hedge parameters including delta, gamma and vega (the effect of the volatility on the option value) of the barrier options. Examples are using an up-and-out foreign currency call against domestic currency put option with exchange rate (F) = 100, strike price (K) = 100, barrier (H) = 115, time to maturity (t) = 6 months, foreign currency interest rate (d) = 5%, domestic currency interest rate (r) = 1%, volatility (σ) = 15%, and κ = 0.5, 1 and 2. These values of κ are consistent with the estimates of κ (which are between 0.1 and 1.2) in Sørensen (1997) for the exchange rates of USD against GBP, DEM and JPY in the period from January 1985 to December 1994. Option values and hedge parameters under the MRL model based on Equation (13) are compared with those of an up-and-out call option under the Black-Scholes model (i.e., κ = 0). β is set to be $-c_2(t)/c_1(t)$ for the initial value of the time-dependent barrier $h(t)$ such that the moving barrier is close to 115 over time. Figure 1 illustrates the movements of the barriers with different κ over option life. It shows that the form of the barrier specified in Equation (11) ensures that the barrier converges to a fixed barrier when κ is reduced to zero. The values of the hedge parameters are measured as finite difference approximations to their continuous time equivalents.

Figure 2 illustrates the option value variations with different mean exchange rates F_0 of the up-and-out calls. When F_0 is below the strike price of 100, the barrier options under the MRL model, especially with high κ, cost less than the barrier option under the Black-Scholes model because the mean-reverting process will push the exchange rate below the strike over time and make the option become out-of-the-money. Conversely, when F_0 is higher than the strike price and will push the exchange rate towards higher values above the strike price during the option life, the option values under the MRL model increase with F_0 and are higher than those under the Black-Scholes model. In the case of strong mean reversion (i.e., κ = 2), the option values under the MRL model decreases when F_0 is higher than 115, where risk of knock-out increases. In the case of κ = 1, when F_0 is higher than 122, the variation of the option values with the underlying price is small. It is because the risk of knock-out and the effect of in-the-moneyness due to the mean-reverting process cancel each other.

Compared with Figure 2, the option values are higher in Figure 3 where the option maturity is reduced to 3 months. The shorter time to maturity reduces the probability of knock-out and thus enhances the values of the up-and-out calls. The effects of the mean-reverting process on the option values in Figure 2 are in general similar to those in Figure 3. In the case of strong mean reversion (i.e., κ = 2), the option values under the MRL model decrease when F_0 is higher than 128, which is above the barrier at 115. This reflects that the effect of the mean reversion on option values weakens with the shorter option maturity.

Dealers need accurate values for the delta and the delta behaviour in order to hedge their option portfolios effectively. The delta variations with the exchange rate *F* in Figure 4 with $F_0 = 100$ and the option maturity of 6 months show two typical features of the up-and-out calls. First, their deltas are smaller than those of the regular option because of the cheaper knock-out option values.[11] Second, the deltas become negative when the underlying price is close to the knock-out barrier. This is due to the fact that the option values decrease when the underlying price moves higher and closer to the barrier. Figure 4 also illustrates that the differences in deltas between the MRL and Black-Scholes models are in general not significant. However, in Figure 5 where the mean exchange rate F_0 is set to be 110, the changes in the deltas with *F* are quite different among the option values under the MRL and Black-Scholes models. For instance, when *F* is 104, the delta under the Black-Scholes model is zero, whereas the deltas under the MRL model with $\kappa = 1$ and 2 are about -0.05 and -0.11, respectively. As the mean exchange rate F_0 is at 110, the mean-reverting process will push the exchange rate at 100 towards higher values and increases the risk of knock-out during the option life. On the other hand, when the options are out-of-the-money, the deltas under the MRL model are higher in values because the mean-reverting process will push the exchange rate towards higher values without significantly increasing the risk of knock-out.

Figure 6 illustrates the variations of gamma with the exchange rate *F* with the options in Figure 5. The gamma is defined as the change in delta per unit of the exchange rate. Similar to the gamma under the Black-Scholes model, the gamma under the MRL model can be negative because of the negative slopes of the delta (see Figure 5) and has the lowest value at -0.025 where *F* = 106. In terms of magnitude, the gamma under the MRL model is in general lower than the gamma under the Black-Scholes model, when *F* is above 103. This reflects that the mean-reverting process reduces the gamma risk of the up-and-out call by pushing the exchange rate to be around the mean exchange rate F_0 at 110. Compared with the Black-Scholes model, the delta under the MRL model is relatively stable for effective hedging due to the lower gamma.

The vega variations in Figure 7 are the percentage differences of the option value with 1% increase in the volatility. The vega of the up-and-out calls turns from positive to negative when the exchange rate is close to the barrier. It is a typical feature of an up-and-out call, where the higher volatility makes the call easier to be knocked out near the barrier. Figure 7 shows that the vega risk in the MRL model is smaller than that in the Black-Scholes model, and the vega decreases with an increase in κ. This observation is consistent with the analysis that the effective volatility $\tilde{\sigma}$ in Equation (19) under the mean-reverting process is less than σ.

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[11] The delta of a regular at-the-money call/put under the Black-Scholes model is about 0.5.

Figure 8 compares the option value variations with different mean exchange rates F_0 of the up-and-out and regular calls. Similar to the price relations between the barrier and regular options under the Black-Scholes model, the values of the up-and-out calls under the MRL model are less than those of the regular calls. Because of no barrier stopping the increase in option values, the option values of the regular calls increase with the mean exchange rate, as F_0 will push the exchange rate towards higher values above the strike price, in particular in the case of strong mean reversion (i.e., $\kappa = 2$). When F_0 is below 102, the regular calls with $\kappa = 2$ cost less than the regular calls with $\kappa = 0.5$ because the stronger mean-reverting process of $\kappa = 2$ will push the exchange rate below the strike over time and make the option become out-of-the-money. This characteristic is similar to that of the up-and-out calls.

In Figure 9, the delta variations with the exchange rate F of the regular calls with $F_0 = 100$ show that their deltas are positive because there is no barrier to reduce the option values when the exchange rate increases. As reflected in the option values in Figure 8, their deltas are higher than those of the up-and-out calls. The deltas of the regular call under strong mean reversion (i.e., $\kappa = 2$) are smaller than those of the regular call with $\kappa = 0.5$ when the exchange rate is higher than 95 because the strong mean-reverting process will push the exchange rate towards the mean level of 100 and thus limits the up-side of the call.

4. SUMMARY

This paper presented a model for valuing currency barrier options when the foreign exchange rate follows a MRL process. The corresponding closed-form solutions for the option valuation with time-dependent barriers are derived. The mean-reverting process keeps the exchange rate in a small range around the mean level over the option life and hence may limit the uncertainty of the option to be knocked out. In the case of an up-and-out call, when the mean level is below the barrier but above the strike price, the risk of the call to be knocked out is reduced and its option value is enhanced compare with the value under the Black-Scholes model. The numerical results show that the mean exchange rate (relative to the current exchange rate and barrier) and the speed of mean reversion have material impact on the valuation of currency barrier options and their hedge parameters.

As mean reversion could be present in foreign-exchange dynamics and the use of currency barrier options for hedging foreign exchange exposures and structuring financial products is popular in the financial market, the findings here could provide an analytical valuation framework for research in currency option pricing. This framework can also be applied to other barrier options in which the underlying assets follow a mean-reverting process. For example, commodity prices such as energy prices seem to exhibit some mean reversion.

APPENDIX A

Regarding a down-and-out call, the solution of Equation (2) at option maturity is

$$\tilde{P}(x,t)=\int_0^\infty dx'\left[G(x,t;x',t=0)-G(x,t;-x',t=0)e^{-\beta x'}\right]\tilde{P}(x',t=0), \qquad \text{(A1)}$$

where

$$\tilde{P}_{call}(x,t=0)=\max\left(He^x-K,0\right), \quad F>h(t),$$
$$\tilde{P}_{call}(x,t=0)=0, \quad F\le h(t). \qquad \text{(A2)}$$

If $h(t)$ is less than or equal to K, the value of a down-and-out call is given by

$$\begin{aligned}P_{do-call}(F,t)&=\tilde{F}e^{-rt}N\left(a+\sqrt{2c_1}\right)-Ke^{-rt}N(a)\\&-\tilde{F}\left(\frac{H}{\tilde{F}}\right)^{2-\beta}e^{\left[(\beta-1)^2-(\beta-1)\right]c_1-rt}N\left(d+\sqrt{2c_1}\right)+K\left(\frac{H}{\tilde{F}}\right)^{-\beta}e^{\beta(\beta-1)c_1-rt}N(d)\end{aligned}. \qquad \text{(A3)}$$

If $h(t)\ge K$, then

$$\begin{aligned}P_{do-call}(F,t)&=\tilde{F}e^{-rt}N\left(b+\sqrt{2c_1}\right)-Ke^{-rt}N(b)\\&-\tilde{F}\left(\frac{H}{\tilde{F}}\right)^{2-\beta}e^{\left[(\beta-1)^2-(\beta-1)\right]c_1-rt}N\left(c+\sqrt{2c_1}\right)+K\left(\frac{H}{\tilde{F}}\right)^{-\beta}e^{\beta(\beta-1)c_1-rt}N(c)\end{aligned}. \qquad \text{(A4)}$$

Because the value of a regular call equals the value of a knock-in call plus the value of a knock-out call, the value of an up-and-in call is given by

$$P_{ui-call}(F,t)=P_{call}(F,t)-P_{uo-call}(F,t), \qquad \text{(A5)}$$

and the value of a down-and-in call is given by

$$P_{di-call}(F,t)=P_{call}(F,t)-P_{do-call}(F,t). \qquad \text{(A6)}$$

The value of a regular call can be obtained by

$$\tilde{P}(x,t)=\int_{-\infty}^\infty dx'G(x,t;x',t=0)\tilde{P}(x',t=0) \qquad \text{(A7)}$$

with the call payoff condition and is equal to

$$P_{call}(F,t)=F^{\exp(-\kappa t)}e^{c_1+c_2-rt}N\left(a+\sqrt{2c_1}\right)-Ke^{-rt}N(a). \qquad \text{(A8)}$$

Barrier put values are derived similarly to barrier call values by using the payoff of $\tilde{P}_{put}(x,t=0)=\max\left(K-He^x,0\right)$, if $F<h(t)$ and zero if $F\ge h(t)$ in Equation (6) for an up-and-out put and if $F>h(t)$ and 0 if $F\le h(t)$ in Equation (A1) for a down-and-out put, respectively. When $h(t)\ge K$, an up-and-out put value is

$$\begin{aligned}P_{uo-put}(F,t)&=Ke^{-rt}N\left(-a-\sqrt{2c_1}\right)-\tilde{F}e^{-rt}N\left(-a-\sqrt{2c_1}\right)\\&-K\left(\frac{H}{\tilde{F}}\right)^{-\beta}e^{\beta(\beta-1)c_1-rt}N(-d)+\tilde{F}\left(\frac{H}{\tilde{F}}\right)^{2-\beta}e^{\left[(\beta-1)^2-(\beta-1)\right]c_1-rt}N\left(-d-\sqrt{2c_1}\right)\end{aligned}. \qquad \text{(A9)}$$

If $h(t)<K$, an up-and-out put value is then

$$P_{uo-put}(F,t) = Ke^{-rt}N(-b) - \tilde{F}e^{-rt}N\left(-b-\sqrt{2c_1}\right)$$
$$-K\left(\frac{H}{\tilde{F}}\right)^{-\beta} e^{\beta(\beta-1)c_1-rt}N(-c) + \tilde{F}\left(\frac{H}{\tilde{F}}\right)^{2-\beta} e^{\left[(\beta-1)^2-(\beta-1)\right]c_1-rt}N\left(-c-\sqrt{2c_1}\right). \quad \text{(A10)}$$

The down-and-out put value is

$$P_{do-put}(F,t) = Ke^{-rt}N(-a) - \tilde{F}e^{-rt}N\left(-a-\sqrt{2c_1}\right)$$
$$-Ke^{-rt}N(-b) + \tilde{F}e^{-rt}N\left(-b-\sqrt{2c_1}\right)$$
$$+K\left(\frac{H}{\tilde{F}}\right)^{-\beta} e^{\beta(\beta-1)c_1-rt}N(-c) - \tilde{F}\left(\frac{H}{\tilde{F}}\right)^{2-\beta} e^{\left[(\beta-1)^2-(\beta-1)\right]c_1-rt}N\left(-c-\sqrt{2c_1}\right). \quad \text{(A11)}$$
$$-K\left(\frac{H}{\tilde{F}}\right)^{-\beta} e^{\beta(\beta-1)c_1-rt}N(-d) + \tilde{F}\left(\frac{H}{\tilde{F}}\right)^{2-\beta} e^{\left[(\beta-1)^2-(\beta-1)\right]c_1-rt}N\left(-d-\sqrt{2c_1}\right)$$

Similar to knock-in calls, the value of an up-and-in put is given by

$$P_{ui-put}(F,t) = P_{put}(F,t) - P_{uo-put}(F,t), \quad \text{(A12)}$$

and the value of a down-and-in put is given by

$$P_{di-put}(F,t) = P_{put}(F,t) - P_{do-put}(F,t). \quad \text{(A13)}$$

The value of a regular put can be obtained by solving Equation (A7) with a put payoff condition, and is equal to

$$P_{put}(F,t) = Ke^{-rt}N(-a) - F^{\exp(-\kappa t)}e^{c_1+c_2-rt}N\left(-a-\sqrt{2c_1}\right). \quad \text{(A14)}$$

APPENDIX B

As specified in Equation (19), $\tilde{\sigma}$ is

$$\tilde{\sigma} = \frac{\sigma}{\sqrt{2\kappa t}} \sqrt{1 - e^{-2\kappa t}} \; . \qquad \text{(B1)}$$

By putting $y = 2\kappa t$ and squaring Equation (B1), it becomes

$$\tilde{\sigma}^2 = \frac{\sigma^2}{y}\left(1 - e^{-y}\right). \qquad \text{(B2)}$$

The derivative of $\tilde{\sigma}^2$ with respect to y is

$$\frac{d\tilde{\sigma}^2}{dy} = -\frac{\sigma^2}{y^2}\left(1 - e^{-y}\right) + \frac{\sigma^2}{y} e^{-y} \; . \qquad \text{(B3)}$$

It is noted that

$$\frac{1}{y^2}\left(1 - e^{-y}\right) = \frac{e^{-y}}{y} \times \frac{1}{y}\left(e^{y} - 1\right). \qquad \text{(B4)}$$

By using Taylor expansion for e^{y}, Equation (B4) becomes

$$\begin{aligned} \frac{1}{y^2}\left(1 - e^{-y}\right) &= \frac{e^{-y}}{y} \times \frac{1}{y}\left(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots - 1\right) \\ &= \frac{e^{-y}}{y}\left(1 + \frac{y}{2!} + \frac{y^2}{3!} + \cdots\right) \qquad , \; y > 0 \, . \\ &> \frac{e^{-y}}{y} \end{aligned} \qquad \text{(B5)}$$

From Equation (B5), because of the inequality in Equation (B5),

$$\frac{d\tilde{\sigma}^2}{dy} < 0 \, . \qquad \text{(B6)}$$

This shows that $\tilde{\sigma}^2$ is a decreasing function with y and thus with κ.

REFERENCES

Anthony, M., & MacDonald, R. (1998). On the mean reverting properties of target zone exchange rates: Some evidence from the ERM. European Economic Review, 42, 1493-523.

Anthony, M., & MacDonald, R. (1999). The width of the band and exchange rate mean reversion: Some further ERM-based results. Journal of International Money and Finance, 18, 411-28.

Ball, C., & Roma, A. (1993). A jump diffusion model for the European Monetary System. Journal of International Money and Finance, 12, 475-492.

Ball, C., & Roma, A. (1994). Target zone modelling and estimation for European Monetary System exchange rates. Journal of Empirical Finance, 1, 385-420.

Black, F., & Scholes, M. (1973). The pricing of options and corporate liability. Journal of Political Economics, 81, 637-654.

Dornbusch, R. (1976). Expectations and exchange rate dynamics. Journal of Political Economics, 84, 1161-1176.

Ekvall, N., Jennergren, L.P., & Näslund, B. (1997). Currency option pricing with mean reversion and uncovered interest parity: A revision of the Garman-Kohlhagen model. European Journal of Operational Research, 100, 41-59.

Garman, M.B., & Kohlhagen, S.W. (1983). Foreign currency option values. Journal of International Money and Finance, 2, 231-237.

Hui, C.H. (1996). One-touch double barrier binary option values. Applied Financial Economics, 6, 343-346.

Hui, C.H. (1997). Time dependent barrier option values. Journal of Futures Markets, 17, 667-688.

Kanas, A. (1998). Testing for a unit root in ERM exchange rates in the presence of structural breaks: Evidence from the boot-strap. Applied Economics Letters, 5, 407-410.

Krugman, P. (1991). Target zones and exchange rate dynamics. Quarterly Journal of Economics, 106, 669-682.

Kunitomo, N., & Ikeda, M. (1992). Pricing options with curved boundaries. Mathematical Finance, 2, 275-298.

Lo, C.F., Lee, H.C., & Hui, C.H. (2003). A simple approach for pricing Black-Scholes barrier options with time-dependent parameters. Quantitative Finance, 3, 98-107.

Lo, C.F., Lee, H.C., & Hui, C.H. (2006). Valuing double barrier options with time-dependent parameters by the method of multiple images. (working paper). Hong Kong: The Chinese University of Hong Kong.

MacDonald, R. (1988). Floating Exchange Rates (p. 293). London: Unwin Hyman.

Merton, R.C. (1973). Theory of rational option pricing. Bell Journal of Economics and Management Science, 4, 141-183.

Nieuwland, F.G.M.C., Verschoor, W.F.C., & Wolff, C.C.P. (1994). Stochastic jumps in EMS exchange rates. Journal of International Money and Finance, 13, 699-727.

Rapisarda, F. (2003). Pricing barriers on underlyings with time-dependent parameters. (working paper). Banca IMI.

Rich, D.R. (1994). The mathematical foundations of barrier option-pricing theory. Advances in Futures and Options Research, 7, 267-311.

Robert, R.O., & Shortland, C.F. (1997). Pricing barrier options with time-dependent coefficients. Mathematical Finance, 7, 83-93.

Rose, A.K., & Svensson, L.E.O. (1994). European exchange rate credibility before the fall. European Economics Review, 38, 1185-1216.

Rubinstein, M., & Reiner, E. (1991). Breaking down the barriers. Risk Magazine, 8, 28-35.

Sørensen, C. (1997). An equilibrium approach to pricing foreign currency options. European Financial Management, 3, 63-84.

Svensson, L.E.O. (1992). An interpretation of recent research on exchange rate target zones. Journal of Economic Perspectives, 6, 119-44.

Svensson, L.E.O. (1993). Assessing target zone credibility. European Economic Review, 37, 763-802.

Figure 1. Barrier level with time to maturity

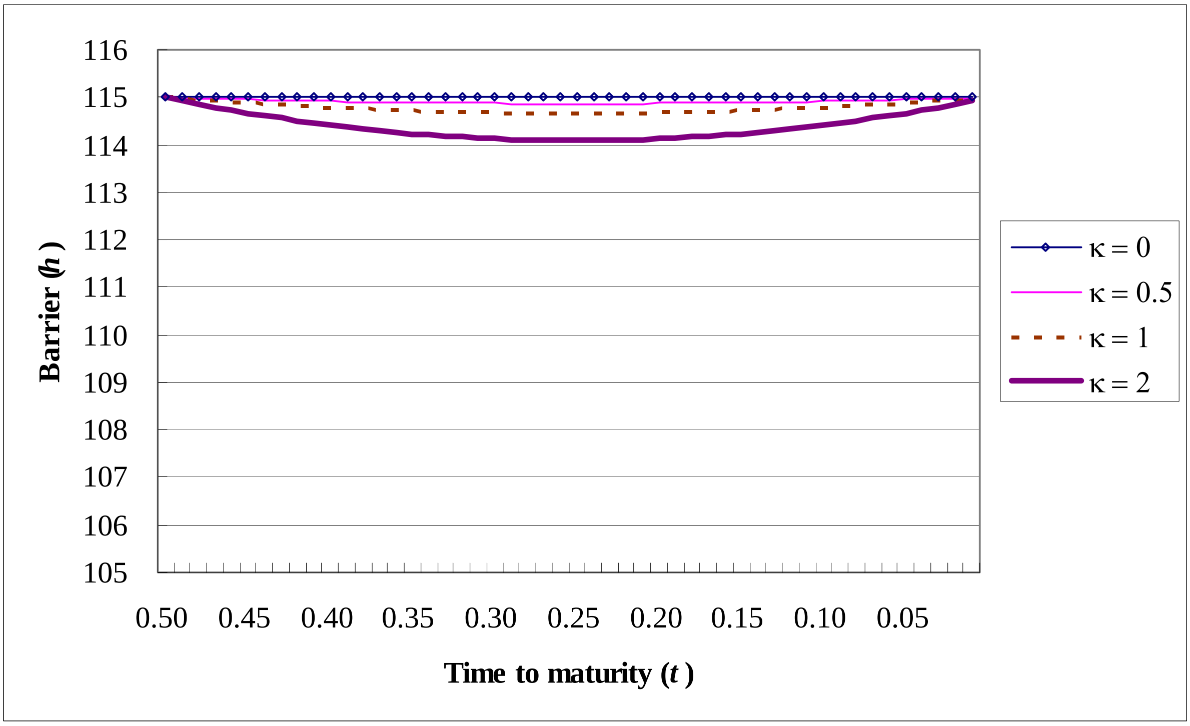

Figure 2. Variation of value of 6-month up-and-out calls with the mean exchange rate

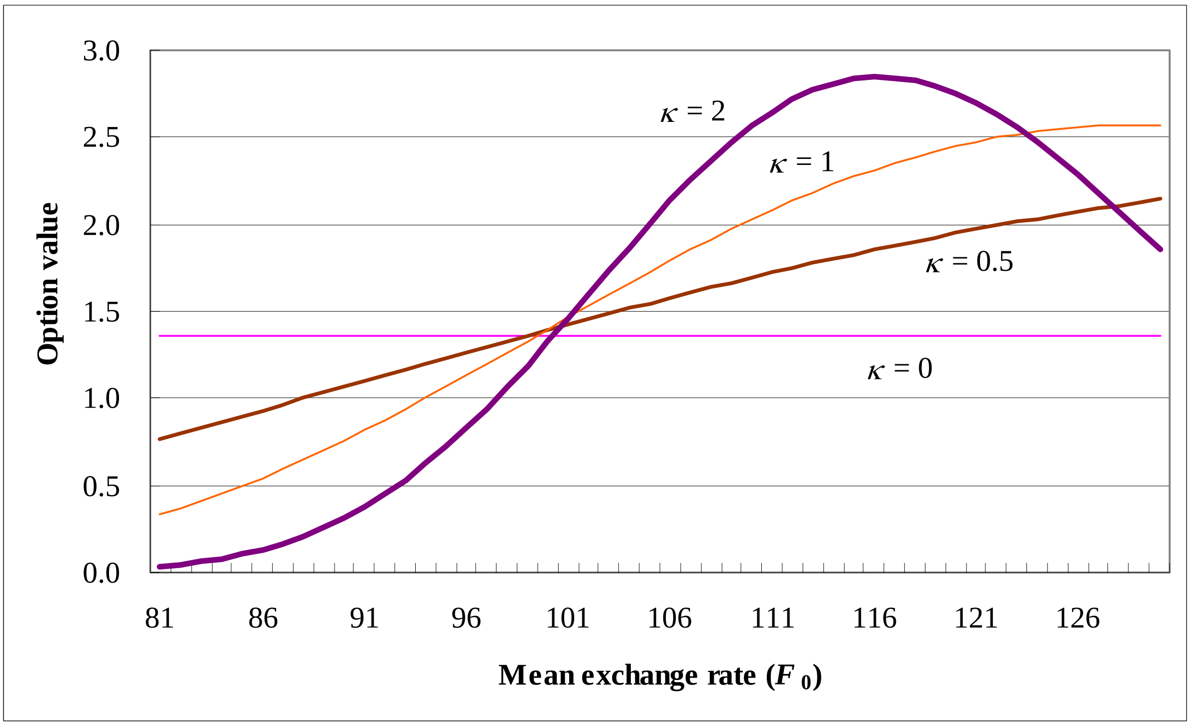

Figure 3. Variation of value of 3-month up-and-out calls with the mean exchange rate

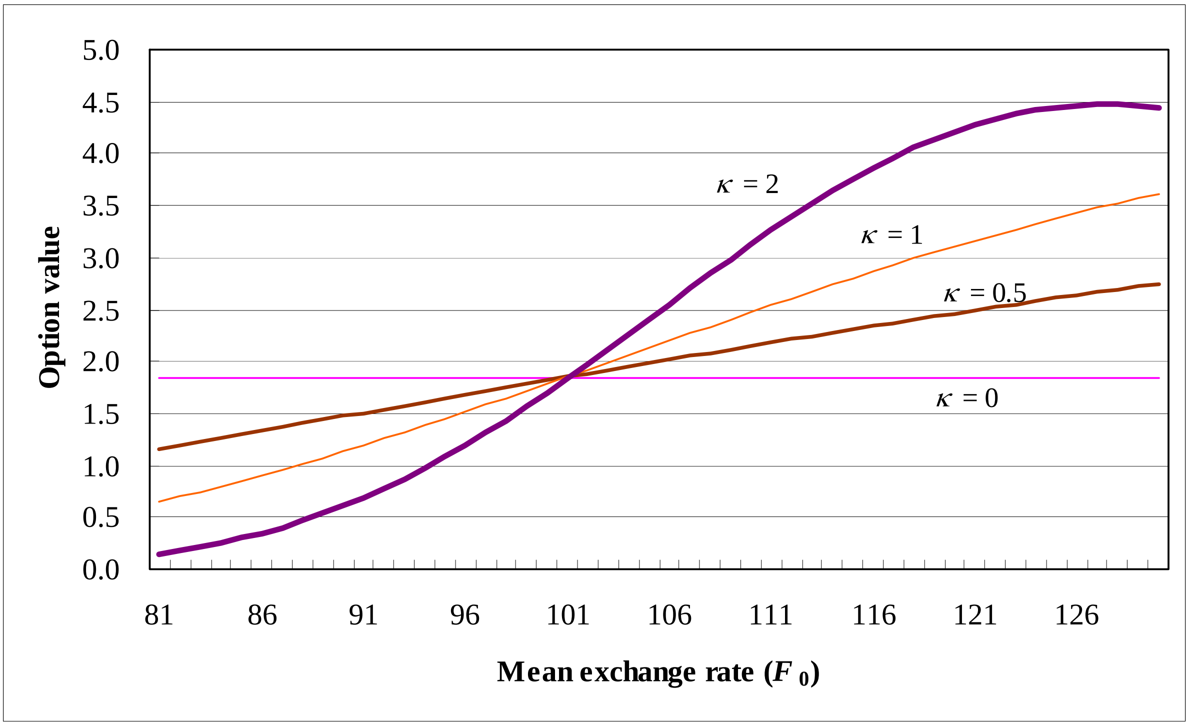

Figure 4. Variation of delta of 6-month up-and-out calls with the exchange rate, where $F_0 = 100$

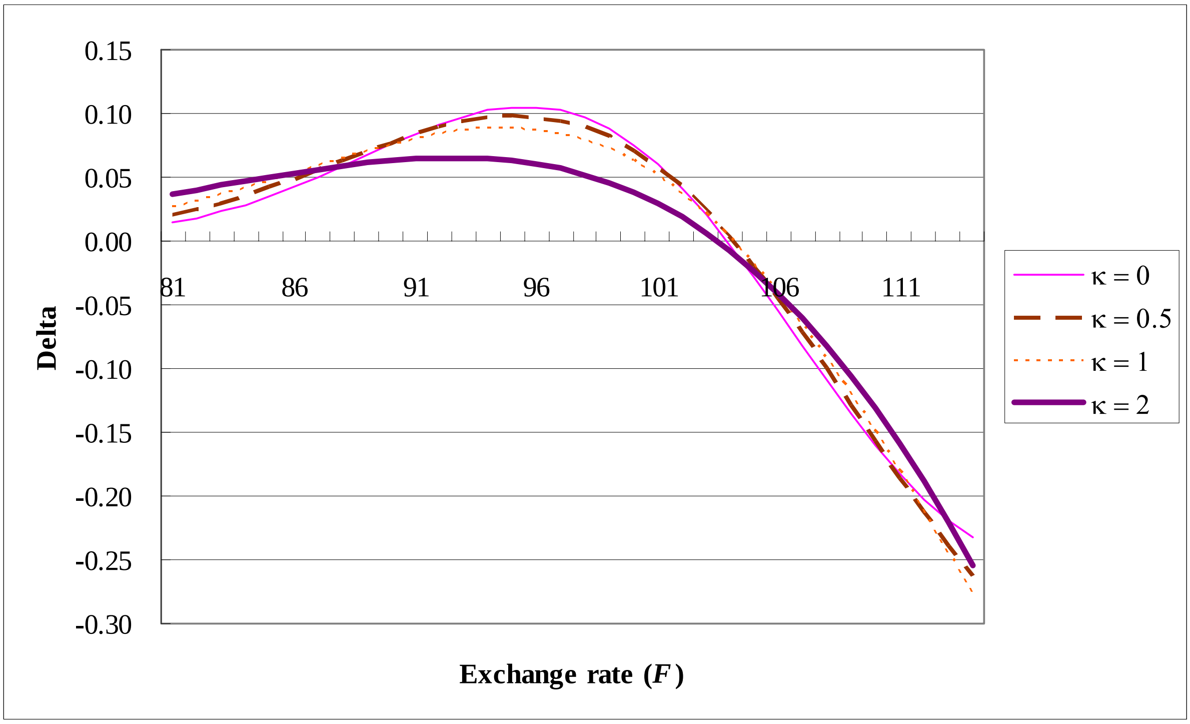

Figure 5. Variation of delta of 6-month up-and-out calls with the exchange rate, where $F_0 = 110$

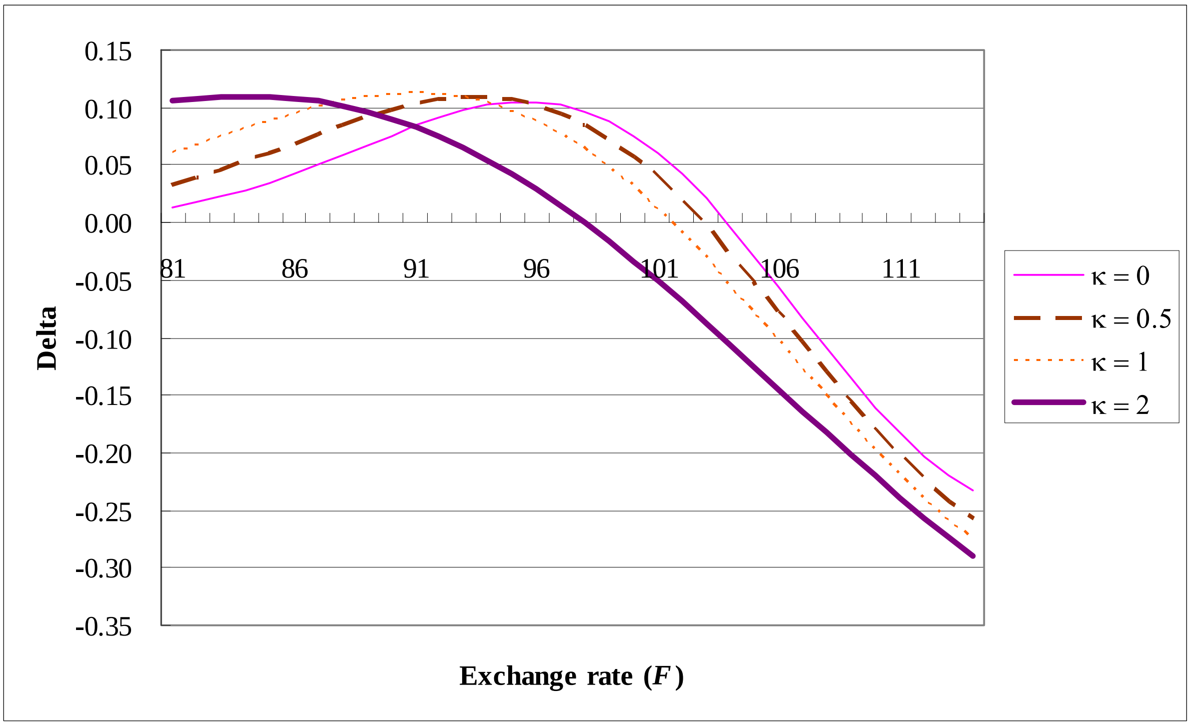

Figure 6. Variation of gamma of 6-month up-and-out calls with the exchange rate, where $F_0 = 110$

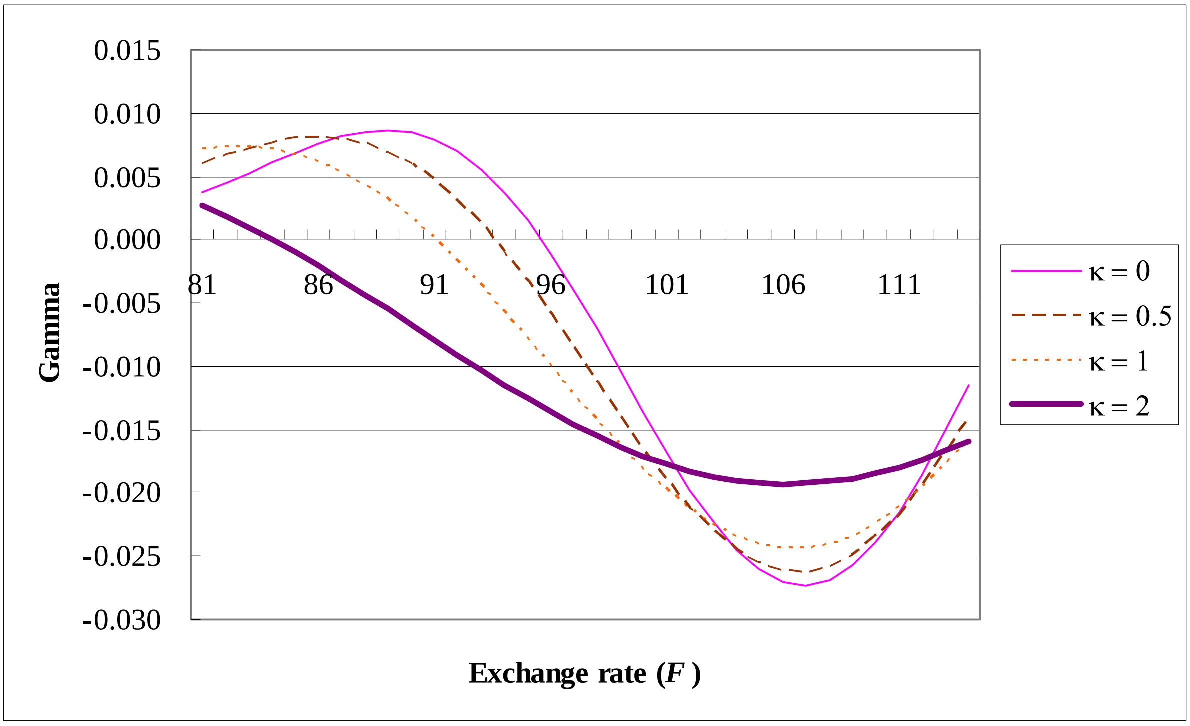

Figure 7. Variation of vega of 6-month up-and-out calls with the exchange rate, where $F_0 = 110$

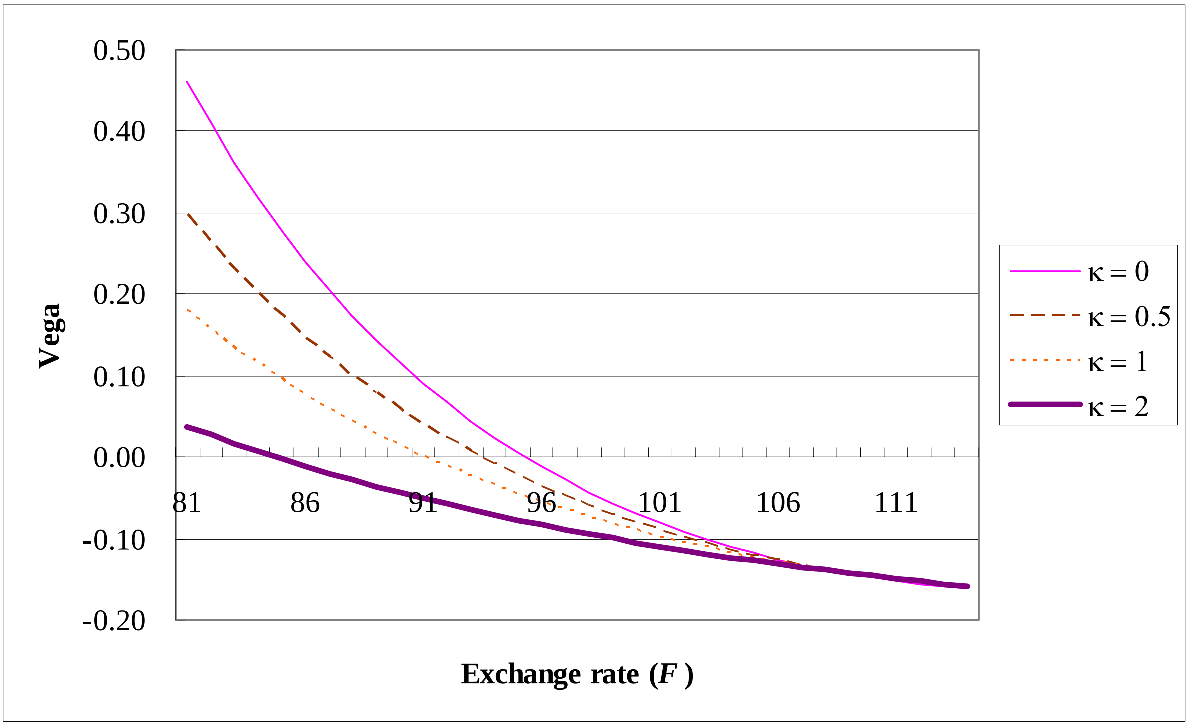

Figure 8. Variation of value of 6-month up-and-out and regular calls with the mean exchange rate

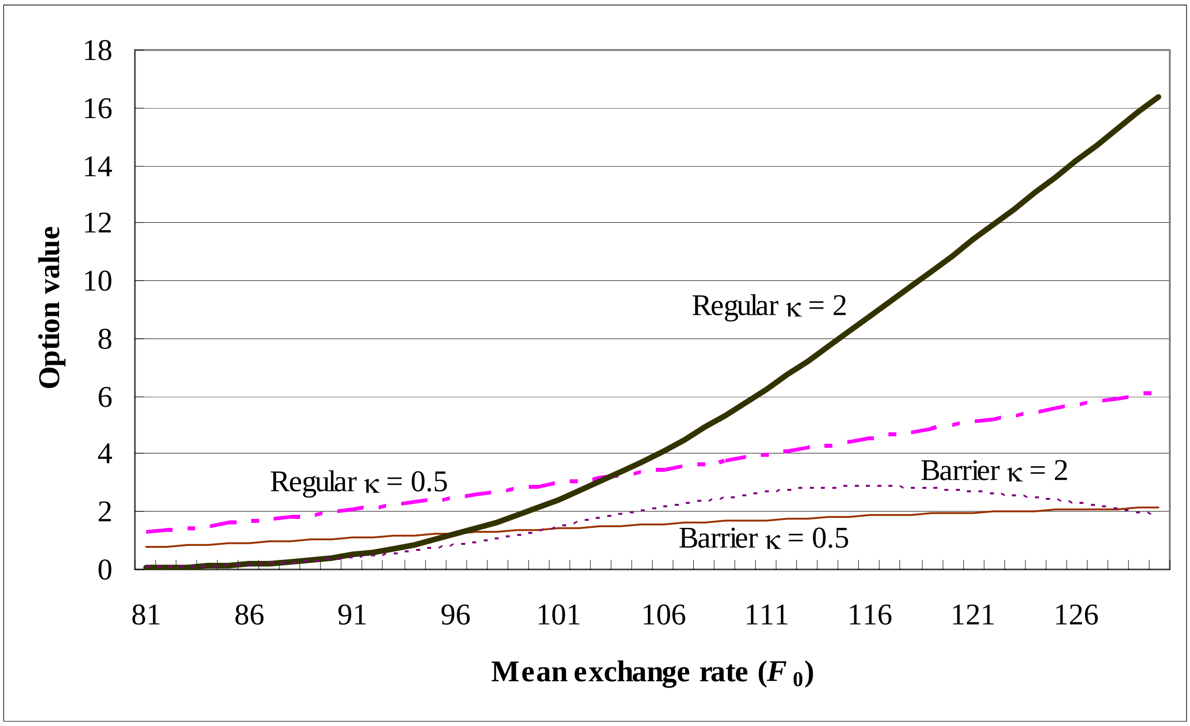

Figure 9. Variation of delta of 6-month up-and-out and regular calls with the exchange rate, where $F_0 = 100$